PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 2, 1980

HOMOGENEOUS MODELS AND DECIDABILITY

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Fix a countable first order structure $\mathscr{A}$ realizing only recursive types. It is known that if $\mathscr{A}$ is prime or saturated then it is decidable iff the set of types it realizes is recursively enumerable. A natural conjecture was that the techniques of proof for those two cases could be combined to produce the result for those $\mathscr{A}$ that are homogeneous. This paper provides a negative answer to that conjecture.

For a complete decidable theory T, let $\{\theta_i \mid i < w\}$ be some fixed effective enumeration of all the formulas of $L(T)$. Then by an *index* for a recursive n-type $\Gamma(x_1, \cdots, x_n)$ of T we mean a natural number e satisfying:

$$\{e\}(i) = \begin{cases} 0 & \text{if} \quad \theta_i \in \Gamma \\ 1 & \text{otherwise} \end{cases}$$

(for notation, see [5]). Also, if Φ is a set of recursive types of T, then a *witness set* A for Φ is a set of natural numbers satisfying:

(1) $\forall n \in A \; \exists \Gamma \in \Phi$ (n is an index for Γ); and

(2) $\forall \Gamma \in \Phi \; \exists n \in A$ (n is an index for Γ).

If Φ is exactly the set of types $\Gamma(x_1, \cdots, x_n)$ realized in some model $\mathscr{A}$ of T satisfying $(x_i \neq x_j) \in \Gamma(x_1, \cdots, x_n)$, $1 \leqq i < j \leqq n$, $n < \omega$ then we also say that A is a witness set for $\mathscr{A}$. Finally, a model $\mathscr{A}$ of T is *decidable* just in case the theory of $(\mathscr{A}, a_i)_{i<\omega}$ is decidable for some indexing $\{a_i \mid i < \omega\}$ of $|\mathscr{A}|$. An undecidable model is a countable model that is not decidable.

Assume now that $\mathscr{C}$ is a prime model. Harrington [2] proved an equivalent version of (by the definitions, if a set of types has a witness set, then those types are recursive):

(*) $\mathscr{C}$ is decidable iff $\mathscr{C}$ has an r.e. witness set.

From a recursion theoretic point of view, the principal device in the proof is a "wait and see" argument. Millar and Morley independently proved that (*) remains true when $\mathscr{C}$ is assumed to be countable and saturated. The principal recursion theoretic technique employed is a finite injury priority argument. Notice that a prime or saturated model is automatically homogeneous, and that any hemogeneous model is uniquely determined, up to isomorphism, by the set of types it realizes. It was therefore very natural that Morley asked whether (*) remained true under just the assumption that $\mathscr{C}$ was countable and homogeneous. This paper provides a negative answer. Interestingly, the construction exploits an "infinite injury". Sufficient con-