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DIMENSION MODULES

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M is called a dimension module if $d(A+B)=d(A)+d(B)-d(A\cap B)$ holds for all submodules A and B of M, where $d(M)$ denotes the Goldie (uniform) dimension of a module M. We characterize these modules as the modules which have no submodules of the form $X\oplus X/Y$ with Y an essential submodule of X. As a test, the structure of a completely decomposable injective dimension module is determined.

A sum $A+B$ of submodules of a module M need not satisfy the usual vector space dimension formula $d(A+B)=d(A)+d(B)-d(A\cap B)$, where $d(M)$ denotes the Goldie dimension of M (that is, $d(M)$ is the number of components in a longest direct sum of submodules contained in M, and is ∞ if no such direct sum exists). This was noted by the authors in [1], where the following substitute formula was proved for arbitrary modules.

THEOREM (*Dimension Formula I*). *Let A and B be submodules of a module M. Let $C=A\cap B$ and let 1_C denote the identity map on C. Let g be a maximal monic extension of 1_C considered as a partial homomorphism from A to B, and let D be the domain of g. Then*

$$d(A+B)=d(A)+d(B)-d(D)+d(D/C)\ .$$

In this paper, we study modules whose submodules satisfy the usual vector space dimension formula itself; these we call *dimension modules*. This class turns out to be somewhat larger than we had originally anticipated. It includes, for instance, all nonsingular modules and all modules whose lattice of submodules is distributive. These examples are obtained in § 1 from a characterization of dimension modules which arises, in turn, from a revision of the dimension formula. In § 2 we show that maximal essential dimension extensions of dimension modules exist. The article concludes in § 3 with a study of injective dimension modules and direct sums of dimension modules. In an appendix, $d(M)$ is compared with the reduced rank $\rho(M)$ of a module M over a right noetherian ring (ρ does satisfy the classical dimension formula).

1. Dimension modules.

We begin with some notation and definitions. All symbols $A, B, M, N, X, Y, \cdots$ indicate modules over an arbitrary ring R. $A\leqq B$ means that A is a submodule of B,