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NEW EXPLICIT FORMULAS FOR THE nTH DERIVATIVE OF COMPOSITE FUNCTIONS

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Dedicated to my mother

The paper consists of four sections, the first of which is an introduction to the problems and a survey of the results known. The second section develops and supplies some new proofs of the fundamental classic formulas deriving another explicit formula for the nth derivative of composite functions. The third section derives new explicit formulas for the nth Lie derivative, i. e., for the nth derivative of composite functions, defined implicitly by the parametric representation $w=g(t)$, $z=f(t)$ and, in particular, for the nth derivative of inverse functions. Compared to the classic formula of Lagrange, the Taylor coefficients of the parametrically given composite functions are here determined by new formulas as explicit functions of the Taylor coefficients of the two component functions. In particular, the respective explicit inverses in the famous class S of regular schlicht functions in the unit disk are found. Moreover, an explicit expression for the substitution of the higher derivatives in Legendre transformations has been given. The fourth section points out the conditions under which all result proved in the previous sections remain valid and are in the real domain. Also, it is noted that the corresponding results remain valid and are for the formal power series.

1. **Einleitung.** Seien die Funktion $w = \rho(z)$ im Gebiet G_z der Ebene z und die Funktion $z = f(t)$ im Gebiet G_t der Ebene t regulär, wobei $G_z = f(G_t)$, dann ist die zusammengesetzte Funktion $w = g(t)$: $=\rho(z)\circ f(t)$, wo die Operation $\circ$ die Substitution $z = f(t)$ bezeichnet, in G_t regulär und ihre erste Ableitung ist in jedem beliebigen Punkt $t \in G_t$ gleich

$$\frac{dw}{dt} = \rho'(z)f'(t) \ . \tag{1}$$

Dem Problem zum Finden expliziter Formeln der $n$te Ableitung $d^nw/dt^n (n \geqq 1)$ sind Abhandlungen vieler Autoren gewidmet. Seit langem ist bemerkt worden, dass bei aufeinanderfolgender Differentiation von (1) nach t durch Induktion folgende Formel für die $n$te Ableitung

$$\frac{d^nw}{dt^n} = \sum_{k=1}^{n} A_{nk}(t)\,\frac{d^kw}{dz^k} \quad (n \geqq 1) \tag{2}$$

erhalten wird, wo die Koeffizienten A_{nk} nicht von der Funktion $\rho(z)$,