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TORSION DIVISORS ON ALGEBRAIC CURVES

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Let D be a divisor on a smooth complete algebraic curve C such that the multiple dD is rationally equivalent to zero for some positive integer d. We may write $D = D_0 - D_\infty$ where D_0 and D_∞ are distinct effective divisors.

The purpose of this note is to prove the

THEOREM. *Assume that C is a curve with general moduli in characteristic zero. For such a divisor D, the cohomology $H^1(C, \mathscr{O}_C(D_0 + D_\infty))$ must be zero (equivalently $|K - D_0 - D_\infty|$ is empty).*

This theorem answers a question which arose in Sevin Recillas' unpublished work which motivated this note. This paper contains simple computational techniques for handling infinitisimal deformations of curves. I hope that these techniques may be useful for solving similar problems.

The theorem is well-known for the case $d = 1$. For instance see Farkas' paper [2]. The recent work [1] contains further biliographic material and related material concerning deformations of mappings of curves into projective spaces.

1. First order deformations defined by principal parts.

Let $\mathscr{L}$ be an invertible sheaf on a smooth curve C. For any point c of C, $\mathrm{Prin}_c(\mathscr{L}) \equiv \mathrm{Rat}(\mathscr{L})/\mathscr{L}_c$ measures the principal (polar) part of the rational sections of $\mathscr{L}$. Let $\mathrm{Prin}(\mathscr{L})$ be the space of principal parts, $\bigoplus \mathrm{Prin}_c(\mathscr{L})$, where c runs through C. For a given principal part $p = (p_c)$ in $\mathrm{Prin}\,(\mathscr{L})$, the support, $\mathrm{supp}(p)$, is the set of c in C with $p_c \neq 0$.

As in [3], we have a natural exact sequence

$$0 \longrightarrow \Gamma(C, \mathscr{L}) \longrightarrow \mathrm{Rat}(\mathscr{L}) \longrightarrow \mathrm{Prin}(\mathscr{L}) \longrightarrow H^1(\mathrm{C}, \mathscr{L}) \longrightarrow 0\ .$$

Let p_c be a principal part in $\mathrm{Prin}_c(\mathscr{L})$ with a pole of order one. By duality the image of p_c in $H^1(C, \mathscr{L})$ is zero if and only if each section of $\Omega_C \otimes \mathscr{L}^{\otimes -1}$ vanishes at c. Consequently, if $H^1(C, \mathscr{L}) \neq 0$ (i.e., there is a nonzero such section), the cohomology class of p_c is nonzero for all but a finite number of points c.

Let $\Theta_C = \Omega_C^{\otimes -1}$ be the sheaf of regular vector fields on C. For any principal part X in $\mathrm{Prin}(\Theta_C)$, we want to define a deformation C_X of C over $T = \mathrm{Spec}(k[\delta])$, where $k[\delta]$ is the ring of dual numbers $k[t]/(t^2)$. First note that, if f is a rational function which is regular at each point of $\mathrm{supp}(X)$, then $X(f)$ makes sense in an obvious way