[WEAKLY] COMPACT OPERATORS AND
DF SPACES

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This is a study of (spaces of) [weakly] compact linear operators with ranges in Fréchet spaces. Characterizations of such operators, extensions and refinements of Schauder's and Gantmaher's Theorems, and results on the approximation property of the space $K(X, Y)$ of compact linear operators are given, together with applications to [weakly] compact operators on function spaces with the strict topology of R. C. Buck. Finally, a new tensor product representation for $K^*(X, Y)$, X and Y Banach, is established, and compact sets of compact operators on Banach spaces are characterized. The main tools are extensions of Grothendieck's DF techniques.

Introduction. This paper is devoted to a study of (spaces of) compact and weakly compact linear operators with ranges in Fréchet spaces. The class of domain spaces is specified to be a class of generalized DF spaces (gDF), which, besides its classical ancestors (and thus all normed spaces), includes the duals of Fréchet spaces under various of the common polar topologies, as well as all function spaces with a strict-like topology as first introduced on spaces of bounded continuous and of bounded holomorphic functions by R. C. Buck [4].

Among the results are an extension and refinement of Schauder's and Gantmaher's Theorems on the [weak] compactness of a linear operator and its adjoint (§3, Theorems 3.1 and 3.2), a new tensor product representation for the space $K_b(X, Y)$ of compact operators and its dual $K^*(X, Y)$, X and Y Banach (§3, Theorem 3.4), or, more generally, X gDF and Y Fréchet (§3, Theorem 3.3), characterizations of operator norm compact sets of compact operators (§4), and a proof of the approximation property for spaces of compact operators (§1, Theorem 1.14).

The principal tools are extensions of Grothendieck's classical DF space techniques to the wider class of gDF spaces (generalized DF): A locally convex space X is gDF, whenever (1) its strong dual is Fréchet, and (2) its topology is localizable on the bounded sets, i.e. linear operators into other locally convex spaces are continuous as soon as their restrictions to the bounded sets are. Generally speaking, "all" DF properties carry over to gDF spaces. The primary object of §1 is to verify this for two of the most fruitful DF properties, for which it has been an open problem. Extending the respective