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RESOLUTION OF AMBIGUITIES IN THE EVALUATION OF CUBIC AND QUARTIC JACOBSTHAL SUMS

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If $p \equiv 1 \pmod{2k}$ is a prime, the Jacobsthal sum $\Phi_k(D)$ is defined by

$$\Phi_k(D) = \sum_{x=1}^{p-1} \left(\frac{x(x^k+D)}{p}\right) \quad (k = 2, 3, \cdots) .$$

It is shown how to evaluate $\Phi_2(D)$ and $\Phi_3(D)$ for any integer D.

1. Introduction. The Jacobsthal sum $\Phi_k(D)$ is defined for primes $p \equiv 1 \pmod{2k}$ by

$$\Phi_k(D) = \sum_{x=1}^{p-1} \left(\frac{x(x^k + D)}{p}\right) , \quad k = 2, 3, \cdots , \tag{1.1}$$

where $(\overline{\ \ p\ })$ is the Legendre symbol, and D is an integer not divisible by p. It is well-known (see for example [8: p. 104]) that

$$\Phi_k(Dm^k) = \left(\frac{m}{p}\right)^{k-1} \Phi_k(D) , \quad m \not\equiv 0 \pmod{p} . \tag{1.2}$$

In this paper, we show how to resolve the sign ambiguities in the evaluations of $\Phi_2(D)$ and $\Phi_3(D)$. (For a discussion of Jacobsthal sums see, for example, [7], [14], [1].)

2. $k = 2$. In this case $p \equiv 1 \pmod 4$ and there are integers a and b such that

$$p = a^2 + b^2 , \quad a \equiv 1 \pmod 4 , \quad b \equiv 0 \pmod 2 , \tag{2.1}$$

with a and $|b|$ unique. Relation (1.2) gives in this case

$$\Phi_2(Dm^2) = \left(\frac{m}{p}\right)\Phi_2(D) , \quad m \not\equiv 0 \pmod{p} , \tag{2.2}$$

so that it suffices to consider $\Phi_2(D)$ for squarefree D. Choosing m such that $m^2 \equiv -1 \pmod{p}$ in (2.2), we have

$$\Phi_2(-D) = (-1)^{(p-1)/4}\Phi_2(D) , \tag{2.3}$$

so that we may take D positive. Jacobsthal [7: pp. 240-241] has evaluated $\Phi_2(1)$. He has shown that

$$\Phi_2(1) = -2a , \tag{2.4}$$

and thus, by (2.2), for any D with $(D/p) = +1$, say $D \equiv E^2 \pmod{p}$,