PACIFIC JOURNAL OF MATHEMATICS
Vol. 100, No. 1, 1982

MAXIMAL GROUPS IN SANDWICH SEMIGROUPS OF BINARY RELATIONS

KAREN CHASE

A sandwich semigroup is given as follows. Let R be an arbitrary but fixed binary relation on a finite set X. For relations A and B on X we say $(a, b) \in A * B$ (the product of A and B) if there are c and d in X such that $(a, c) \in A$, $(c, d) \in R$ and $(d, b) \in B$. This semigroup is denoted $B_X(R)$. In this paper we study maximal groups in $B_X(R)$ for various classes of R.

Sandwich semigroups of binary relations were introduced in [2]. These semigroups arise naturally in automata theory, and their role in automata theory is studied in [3]. Montague and Plemmons [5] have shown that given a finite group G there is some set X such that G is a maximal group in B_X, the usual semigroup of binary relations. We show there are classes of R for which this result holds and others for which it does not hold.

If R is a relation and E is a nonzero idempotent in $B_X(R)$, then we write $G_E(R)$ for the maximal group determined by E and call E an R-idempotent. In § 1 we give a class of relations for which $G_E(R)$ is trivial for any relation R in this class and any R-idempotent E. In § 2 we produce a class of relations for which the Montague-Plemmons result holds. That is, any finite group G arises as a maximal group for some X and some relation R in this class. Finally, in § 3 we show there is a class of relations for which some but not all finite groups arise.

Throughout we use Boolean matrix representation for relations. That is, if R is a relation over X where $|X| = n$, then R is represented by an $n \times n$ matrix where the (i, j) entry is a 1 if (x_i, x_j) is in R and 0 otherwise. These matrices are multiplied using Boolean arithmetic.

This paper is part of a Ph. D. thesis prepared under the direction of C. J. Maxson whom I wish to thank for his guidance and many helpful suggestions.

1. $B_X(R)$ containing only trivial groups.

Let Γ be the collection of (nonzero) matrices with the property that all nonzero columns are the same. For R in Γ it is easy to see that if the (i, j) entry of R is zero then either row i or column j of R is zero. The following theorem characterizes R-idempotents for any R in Γ and shows that $G_E(R)$ is trivial for any R in Γ and any R-idempotent E.