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POLYNOMIAL FORMS ON AFFINE MANIFOLDS

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An affine manifold is a differentiable manifold without boundary together with a maximal atlas of coordinate charts such that all coordinate changes extend to affine automorphisms of R^n. These distinguished charts are called affine coordinate systems.

Throughout this paper M denotes a connected affine manifold of dimension $n \geqq 1$. We write E for R^n.

A tensor (field) on M is called *polynomial* if in all affine coordinate systems its coefficients are polynomial functions in n variables. In particular a real-valued function on M may be polynomial.

It is unknown whether there exists any compact affine manifold admitting a nonconstant polynomial function. The main purpose of this paper is to prove that for certain classes of affine manifolds there is no such function. These results are then applied to demonstrate that certain polynomial forms must also vanish. For related results, see Fried, Goldman, and Hirsch [2], Fried [1], [6], and [5].

1. Development, holonomy, and polynomial tensors. Let $p: \tilde{M} \to M$ be a universal covering space. There is an immersion $D: \tilde{M} \to E$, called the developing map, with the following properties (see e.g., [2]):

(1) D is affine, i.e., in affine coordinates D appears as an affine map;

(2) D is unique up to composition with an affine automorphism of E.

We call $D(\tilde{M})$ the *developing image*.

Let π denote the group of deck transformations of $\tilde{M}$. It follows from (2) that there is a homomorphism $\alpha: \pi \to \mathrm{Aff}(E)$, the group of affine automorphisms of E, such that D is equivariant respecting α, that is:

$$D \circ g = \alpha(g) \circ D \quad \text{for all } g \in \pi .$$

We call α the *affine holonomy*. The composition

$$\lambda: \pi \xrightarrow{\alpha} \mathrm{Aff}(E) \xrightarrow{\beta} GL(E) ,$$

where β is the natural homomorphism, is called the *linear holonomy*.

If $\alpha(\pi)$ fixes a point $p \in E$ then M is called a *radiant* manifold. In this case we can compose D with translation by $-p$ to obtain a new developing map whose corresponding affine holonomy fixes