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REMARKS ON NONLINEAR CONTRACTIONS

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Throughout this paper, we assume that K is strongly normal, that $P = \{d(x, y); x, y \in X\}$, that $\bar{P}$ denotes the weak closure of P, and that $P_1 = \{z; z \in \bar{P}$ and $z \neq \mathcal{O}\}$. The main result of this paper is the following.

Let (X, d) be a nonempty K-complete metric space, and let S, T be mappings of X into itself satisfying (1) and (2).

$$\phi(d(Sx, Ty)) \leqq d(x, y)\,, \quad x \neq y \in X\,, \tag{1}$$

$$\phi(t) > t \quad \textbf{for any} \quad t \in P_1\,, \tag{2}$$

where $\phi: P_1 \to K$ is lower semicontinuous on P_1.

Then exactly one of the following three statements holds:

(a) S and T have a common fixed point, which is the only periodic point for both S and T;

(b) There exist a point $x_0 \in X$ and an integer $p > 1$ such that $Sx_0 = x_0 = T^p x_0$ and $Tx_0 \neq x_0$;

(c) There exist a point $y_0 \in X$ and an integer $q > 1$ such that $S^q y_0 = y_0 = Ty_0$ and $Sy_0 \neq y_0$.

Recently, J. Eisenfeld and V. Lakshmikantham [6, 7, 8], J. C. Bolen and B. B. Williams [1], S. Heikkila and S. Seikkala [9, 10], K. J. Chung [3, 4], M. Kwapisz [12] J. Wazewski [16] proved some fixed point theorems in abstract cones which extend and generalize many known results. In this paper, we extend some main results of A. Meir and E. Keeler [14] and C. L. Yen and K. J. Chung [17] to cone-valued metric spaces.

(I). **Definitions.** Let E be a normed space. A set $K \subset E$ is said to be a cone if (i) K is closed (ii) if $u, v \in K$ then $\alpha u + \tau v \in K$ for all $\alpha, \tau \geqq 0$, (iii) $K \cap (-K) = \{\mathcal{O}\}$ where $\mathcal{O}$ is the zero of the space E, and (iv) $K^0 \neq \phi$ where K^0 is the interior of K. We say $u \geqq v$ if and only if $u - v \in K$, and $u > v$ if and only if $u - v \in K$ and $u \neq v$. The cone K is said to be strongly normal if there is a $\delta > 0$ such that if $z = \sum_{i=1}^n b_i x_i$, $x_i \in K$, $\|x_i\| = 1$, $b_i \geqq 0$, $\sum_{i=1}^n b_i = 1$, implies $\|z\| > \delta$. The cone K is said to be normal if there is a $\delta > 0$ such that $\|f_1 + f_2\| > \delta$ for $f_1, f_2 \in K$ and $\|f_1\| = \|f_2\| = 1$. The norm in E is said to be semimonotone if there is a numerical constant M such that $\mathcal{O} \leqq x \leqq y$ implies $\|x\| \leqq M\|y\|$ (where the constant M does not depend on x and y).

Let X be a set and K a cone. A function $d: X \times X \to K$ is said to be a K-metric on X if and only if (i) $d(x, y) = d(y, x)$, (ii) $d(x, y) = \mathcal{O}$ if and only if $x = y$, and (iii) $d(x, y) \leqq d(x, z) + d(z, y)$.