PACIFIC JOURNAL OF MATHEMATICS
Vol. 101, No. 2, 1982

FIXED POINTS ON FLAG MANIFOLDS

HENRY H. GLOVER AND WILLIAM D. HOMER

When K is $\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$, let $U_K(n)$ denote the group of $n \times n$ orthogonal, unitary, or symplectic matrices, respectively. If G is a closed connected subgroup of $U_K(n)$ of maximal rank, then it is conjugate to a subgroup of the form $U_K(n_1) \times U_K(n_2) \times \cdots \times U_K(n_k)$. A simple condition on the integers n_i is shown to be necessary for $U_K(n)/G$ to have the fixed point property (that every self map has a fixed point). It is conjectured that this condition is also sufficient, and a proof is given for some cases.

For a partition $n = n_1 + n_2 + \cdots + n_k$ of a positive integer n and $K = \boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$, the corresponding generalized flag manifold $U_K(n)/(U_K(n_1) \times \cdots \times U_K(n_k))$ will be denoted $KM(n_1, \cdots, n_k)$. We conjecture that $KM(n_1, \cdots, n_k)$ has the fixed point property if and only if $n_1, \cdots, n_k$ are distinct integers and, when $K = \boldsymbol{R}$ or $\boldsymbol{C}$, at most one is odd. We prove that this condition is necessary and that it is sufficient, in addition to previously known cases, for the manifolds $KM(1, n_2, n_3)$ where n_3 is large relative to n_2 (and, when $K = \boldsymbol{R}$, in some other cases as well).

THEOREM 1. *If $KM(n_1, n_2, \cdots, n_k)$ has the fixed point property, then $n_1, \cdots, n_k$ are distinct integers and, if $K = \boldsymbol{R}$ or $\boldsymbol{C}$, at most one is odd.*

Proof. We can regard $M = \boldsymbol{C}M(n_1, \cdots, n_k)$ as the space of orthogonal direct sum decompositions $\boldsymbol{C}^n = V_1 \oplus \cdots \oplus V_k$, where V_m has dimension n_m over $\boldsymbol{C}$. If $n_r = n_s$, interchanging the rth and sth summands defines a fixed point free self map of M.

For the rest of the proof, we define a conjugate linear transformation J of $\boldsymbol{C}^n$ and consider the associated self map f of M, which takes $V_1 \oplus \cdots \oplus V_k$ to $JV_1 \oplus \cdots \oplus JV_k$. If $n = 2m$, we regard $\boldsymbol{C}^n$ as $\boldsymbol{H}^m$ and take J to be multiplication by the quaternion j. Any subspace of $\boldsymbol{C}^n$ invariant under J has the structure of a vector space over $\boldsymbol{H}$ and so has even dimension as a vector space over $\boldsymbol{C}$. Thus if at least one (and so necessarily at least two) of the integers $n_1, \cdots, n_k$ is odd, f has no fixed points.

If $n = 2m + 1$, we write $\boldsymbol{C}^n = \boldsymbol{H}^m \oplus \boldsymbol{C}$ and take J to be multiplication by j on the first summand and complex conjugation on the second. If W is a subspace of $\boldsymbol{C}^n$ which is invariant under J, then its projection onto the first summand is invariant under multiplication by j and so has even dimension over $\boldsymbol{C}$. Hence each odd