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DUALITY AND COHOMOLOGY FOR ONE RELATOR GROUPS

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1. Introduction. Let G be a group having a one relator presentation and some fundamental integral class $[G] \in H_2(G)$. The object of this paper is to study the cap product homomorphism $[G] \cap \cdot$: $H^i(G; A) \rightarrow H_{2-i}(G; \bar{A})$ where A is a left G module and $\bar{A}$ is the right G module identified with A as an abelian group and whose scalar multiplication is given by $ag = g^{-1}a$ for $a \in A$, $g \in G$. If this homomorphism is an isomorphism we say that G satisfies *Poincaré duality* with respect to A.

For example consider the fundamental group G of an orientable surface M. In this case the homomorphism $[G] \cap \cdot$ is an isomorphism for all G modules A. Such a group is said to satisfy *Poincaré duality.* Recently Müller [11, 12] has shown that a one relator group satisfying Poincaré duality over A for all G modules A is isomorphic to the fundamental group of some orientable surface; thus answering a question of Johnson and Wall in [9]. Actually Müller's result is stronger than this since it applies to a larger class of groups than one relator groups. However, we will restrict our attention to one relator groups and study duality for fixed coefficients A.

In § 2 various preliminary work relating Fox derivatives and Magnus expansions is given and in § 3 there are some results for $\boldsymbol{Z}$ coefficients. In particular Theorem 3.4 proves that any group satisfying Poincaré duality over the integers has a presentation of the form $\{x_1, \cdots, x_{2g} \mid [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]W = 1\}$ where W lies in the third term of the lower central series of the free group on $x_1, \cdots, x_{2g}$. Note that if $W = 1$ then the presentation reduces to that of a surface group. This result has been proved independently by Ratcliffe, [15].

In § 4 an explicit formula for the homomorphism $[G] \cap \cdot$ on the chain level is given in terms of a Hessian matrix $\partial_i(\overline{\partial_j V})$ of Fox derivatives, where V is the relator.

Using the theory developed in this paper and results from [16] it is routine to verify the claims made in the following examples.

EXAMPLE. The group $G = \{x_1, x_2 \mid [x_1, x_2][x_2, [x_2, x_1]] = 1\}$ satisfies Poincaré duality over $\boldsymbol{Z}$. Now let A be the Laurent polynomial ring $\boldsymbol{Z}[\boldsymbol{Z}]$ on the generator t with the G module structure induced from the homomorphism $\phi: G \rightarrow \boldsymbol{Z}[t]$ defined by $\phi(x_1) = 1$, $\phi(x_2) = t$. If G were to satisfy Poincaré duality over A then it would be true that