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ESSENTIAL TORI IN 4-MANIFOLD BOUNDARIES

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The central result is an analogue for four manifolds of the loop theorem, in which, with suitable restrictions on π_2, essential loops in the boundary are replaced by essential tori.

A map f: $T^2 \to X$ of a torus into a space is called *essential* if the induced map $f_\#$: $\pi_1(T^2) \to \pi_1(X)$ is injective [**Jo**]. A map which is not essential is called *inessential*. If X is a 3-manifold and f is an essential imbedding then $f(T^2)$ is *incompressible* in X. The question which we examine here is the following: Suppose M is a 4-manifold and there is an essential map f: $T^2 \to \partial M$ which is inessential in M. Is there an incompressible torus in ∂M which is inessential in M?

The question may be viewed as a 4-dimensional analogue of the loop theorem of Papakyriakopoulos [**Pa**], which says that if there is an essential map of a circle to the boundary of a 3-manifold which is inessential in the 3-manifold, then an imbedding with this property exists. There are, however, two points to keep in mind about this comparison. In the classical case, the usefulness of the theorem is greatly enhanced by combining it with Dehn's lemma [**St**], for which no good 4-dimensional analogue has yet been found. Secondly, a map f: $S^1 \to X$ is called essential if $f_\#$: $\pi_1(S^1) \to \pi_1(X)$ is merely non-trivial, not necessarily injective. Thus the meaning of the word "essential" has come to have slightly different meanings, depending on whether the domain of the map is a circle or a torus. However, in the former case, if the range X is a surface or an orientable, sufficiently large 3-manifold the notions coincide, for $\pi_1(X)$ has no torsion.

Before attempting to answer the question, consider two classes of counterexamples. First, let T be any orientable surface with $\chi(T) < 0$, and f: $S^1 \to T$ be a map not homotopic to a multiple of any imbedded loop. Then $\bar{f} = f \times \mathrm{id}_{S^1}$: $S^1 \times S^1 \to T \times S^1 \times 0 \subset T \times S^1 \times I$ is an essential torus in $T \times S^1 \times I$. Let Q be the 4-manifold obtained by adding a 2-handle to $T \times S^1 \times I$ on an imbedded loop in $T \times S^1 \times 1$ homotopic to $f(S^1) \times$ (point). Then $\bar{f}$ is essential in $T \times S^1 \times 0$, inessential in Q, yet any incompressible torus in $T \times S^1 \times 0$ remains essential in Q. Notice that the addition of the 2-handle generates a non-trivial element of π_2, just as adding a 2-disk across the meridian of a torus does.

A more subtle counterexample is the following: Let T be a closed orientable surface with $\chi(T) \leq 0$ and φ: $T \to T$ be a periodic automorphism of period p which is free except at a finite collection of points