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TOPOLOGICAL SPHERICAL SPACE FORM PROBLEM III: DIMENSIONAL BOUNDS AND SMOOTHING

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In the two earlier papers in this series we showed, that if the finite group π has period $2d$ in cohomology, and if for all primes p a subgroup of order $2p$ is cyclic, then there exists a free topological action by π on a sphere of dimension S^{2nd-1}, for some positive integer n. Two questions remained open, namely whether there also existed smooth actions, and whether n could be taken equal to one. In this short paper we prove that there exists a free smooth action of π on $S^{2e(\pi)-1}$, the sphere with the standard differentiable structure. Here $e(\pi)$ is the Artin-Lam induction exponent, that is, the least positive integer such that $e(\pi)1$ belongs to the ideal of the rational representation ring, generated by representations induced from cyclic subgroups. It turns out that $e(\pi) = d(\pi)$ or $2d(\pi)$, and that our result is geometrically the best possible, except for one class of groups.

Our theorem can actually be extracted from various papers already available, chiefly [**3**], [**4**], [**10**] and [**12**], but given its interest, we think it worthwhile to publish this guide to the material separately.

The outline of the argument is as follows. Starting with a periodic projective resolution for the cohomology of π, see [**8**], we choose the chain homotopy type so that, restricted to a hyperelementary subgroup ρ, the resolution is equivalent to a free resolution of period $2e(\pi)$. It now follows that the finiteness obstruction for the π-resolution vanishes, and that there is a finite geometric realisation, $Y(\pi)$. Following [**3**] we construct a smooth normal invariant, and show that the surgery obstruction to replacing $Y(\pi)$ by a homotopy equivalent smooth manifold vanishes. By variation inside the orbit of this manifold under the action of $L_{2e(\pi)}(\mathbf{Z}\pi)$ on $\mathcal{S}_0(Y(\pi))$ we show that the universal cover $Y(1)$ may be taken to be the standard sphere. As in the construction of the complex $Y(\pi)$, in proving the existence of a homotopy smoothing we reduce technical problems by the principle of induction to hyperelementary subgroups. These are necessarily either metacyclic or split extensions of a cyclic group of odd order by a binary dihedral group $D^*_{2^k}$. Once dihedral subgroups D_{2p} are excluded, such a 2-hyperelementary subgroup admits a fixed point free representation of real degree four or eight, see [**13**, p. 168 and 204].