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RANK OF POSITIVE MATRIX MEASURES

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Let L be a selfadjoint operator in a separable Hilbert space. Here we define a concept of rank for positive matrix measures from which the spectral multiplicity of a point in the spectrum of L may be determined. In the process, a diagonalization procedure for positive matrix measures is constructed, connecting the concept of a spectral matrix to the abstract measures of a spectral representation.

The definitions and theorems appearing in paragraph 1–6 are taken directly from the article of Rosenberg [3]. They establish background material essential to the article and serve to familiarize the reader with typical manipulations of positive matrix measures.

1. DEFINITION. Let (φ_{ij}) be a complex matrix valued function on R and ν a non-negative real valued measure on the Borel subsets $\mathfrak{B}$ of the real line. If for each i and j φ_{ij} is $\mathfrak{B}$-measurable and integrable with respect to ν then we say $(\varphi_{ij}) \in \mathfrak{L}(R, \nu)$ and $\int(\varphi_{ij})\, d\nu = (\int \varphi_{ij}\, d\nu)$.

2. Let (ρ_{ij}) be an $n \times n$ non-negative definite hermitian-matrix valued function defined on the bounded Borel subsets of R where each entry function ρ_{ij} is countably additive on $\mathfrak{B}$. The matrix (ρ_{ij}) is called a positive matrix measure. Each ρ_{ii} is a non-negative real valued measure, and each ρ_{ij} for $i \neq j$ is a complex valued measure. From this and the fact that for a non-negative hermitian matrix $H, (0) \leq H \leq (\operatorname{tr} H)I$ where I is the identity matrix and tr denotes trace it follows that each ρ_{ij} is absolutely continuous with respect to the positive measure $\rho = \operatorname{tr}(\rho_{ij}) = \sum_{i=1}^{n} \rho_{ii}$. The Radon-Nikodym derivatives $d\rho_{ij}/d\rho$ are thus well defined up to sets of zero ρ-measure.

3. DEFINITION. The matrix function $(m_{ij}(\lambda)) = (d\rho_{ij}/d\rho)$ will be called the *trace derivative* of (ρ_{ij}). For any measure μ such that $\rho \ll \mu$, $(d\rho_{ij}/d\mu) = (m_{ij})d\rho/d\mu$ will be called the *μ-derivative* of (ρ_{ij}).

4. FACTS. (a) $(m_{ij}(\lambda))$ is $\mathfrak{B}$-measurable and integrable with respect to $\rho = \operatorname{tr}(\rho_{ij})$ and

$$\int_A \big(m_{ij}(\lambda)\big)\, d\rho = \big(\rho_{ij}(A)\big) \quad \text{for } A \in \mathfrak{B}.$$