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ON THE HOLOMORPHY OF MAPS FROM A COMPLEX TO A REAL MANIFOLD

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Let $f\colon X \to Y$ be a C^1-submersion from a complex manifold X to a real C^1-manifold Y. One main object of this paper is to prove two sets of necessary and sufficient conditions which will guarantee that Y can be equipped with a complex structure making f holomorphic. We provide a generic counterexample to show the essential nature of the conditions we establish.

The second set of conditions (Theorem 3) apply even when X is a complex Banach manifold. This theorem is then used to prove the existence of the natural complex structure on the Teichmuller spaces of Riemann surfaces of finite type.

We shall start by giving some necessary conditions in Lemmas 1, 2, 3.

LEMMA 1. *If $f\colon X^m \to Y^d$ is a surjective holomorphic submersion between complex manifolds of dimensions m and d respectively then the fibers of f (i.e. the sets $f^{-1}(y)$, $y \in Y$), are (closed) complex submanifolds in X of dimension $(m-d)$.*

Proof. This is immediate from the implicit function theorem in the holomorphic category.

Let X^m be an m-dimensional complex manifold. We let $\mathrm{Gr}_{(m-d)}(TX)$ denote the Grassmann bundle of $(m-d)$-dimensional complex subspaces in the tangent bundle TX of X. The total space $\mathrm{Gr}_{(m-d)}(TX)$ inherits a natural complex structure from X.

DEFINITION 1. An $(m-d)$-dimensional *distribution* on X is a section of the $\mathrm{Gr}_{(m-d)}(TX)$ bundle over X. We say the *distribution is analytic* if the section is an analytic function.

REMARK. Note that the distribution is analytic if and only if it can be spanned locally by $(m-d)$ linearly independent analytic vector fields.

Let $f\colon X \to Y$ be a C^1-submersion from a complex manifold X onto a real C^1-manifold Y. Then if $y \in Y$ and $x \in f^{-1}(y)$, the differential of f at x $d_x f\colon T_x X \to T_y Y$ is a surjective linear map. If $\ker d_x f$ is a complex