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INVARIANTS OF THE HEAT EQUATION

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Let M be a compact Riemannian manifold without boundary and let P: $C^\infty V \to C^\infty V$ be a self-adjoint elliptic differential operator with positive definite leading symbol. The asymptotics of the heat equation $\mathrm{Tr}(\exp(-tP))$ as $t \to 0^+$ are spectral invariants given by local formulas in the jets of the total symbol of P. Let $A(x)$ and $B(x)$ be polynomials where the degree of B is positive and the leading coefficient is positive. The asymptotics of $\mathrm{Tr}(A(P)\exp(-tB(P)))$ can be expressed linearly in terms of the asymptotics of $\mathrm{Tr}(\exp(-tP))$. Thus no new spectral information is contained in these more general expressions. We also show the asymptotics of the heat equation are generically non-zero. If one relaxes the condition that the leading symbol of P be definite, the asymptotics of $\mathrm{Tr}(\exp(-tP^2))$ and $\mathrm{Tr}(P\exp(-tP^2))$ form a spanning set of invariants. These are related to the zeta and eta functions using the Mellin transform, and a similar non-vanishing result holds except for the single invariant giving the residue of eta at $s = 0$ which vanishes identically.

1. Introduction. Let M be a compact Riemannian manifold of dimension m without boundary. Let P: $C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic differential operator of order $u > 0$. Let $p(x, \xi)$ be the leading symbol of P for $x \in M$ and $\xi \in T^*M_x$. We suppose $p(x, \xi)$ is a positive definite matrix for $\xi \neq 0$; this implies u is even. Let $(\lambda_\nu, \theta_\nu)_{\nu=1}^\infty$ be a spectral resolution of P into a complete orthonormal basis θ_ν for $L^2(M)$ of eigensections so $P\theta_\nu = \lambda_\nu\theta_\nu$. The spectrum of P is bounded from below and the eigenvalues tend towards ∞ as $\nu \to \infty$. Order the eigenvalues so $\lambda_1 \le \lambda_2 \le \cdots$. If $t > 0$, $\exp(-tP)$ is an infinitely smoothing operator with smooth kernel

$$K(x, y, \exp(-tP)) = \sum_\nu \exp(-t\lambda_\nu)\theta_\nu(x) \otimes \theta_\nu(y)$$

where $\theta_\nu(x) \otimes \theta_\nu(y)$ is regarded as an endomorphism from the fiber of the bundle at y to the fiber at x.

On the diagonal, there is an asymptotic expansion of the form:

$$K(x, x, \exp(-tP)) \simeq \sum_{j=0}^{m} t^{(n-m)/u} e_n(x, \exp(-tP)), \quad t \to O^+$$

where $e_n = 0$ if n is odd. In the literature, this sum is often reindexed since $e_1 = e_3 = \cdots = e_{2n+1} = 0$. We shall not adopt this convention as it would lead to difficulties in what follows when considering more general invariants.