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ON SPARSELY TOTIENT NUMBERS

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Let $\varphi(n)$ denote Euler's totient function, defined for $n > 1$ by

$$\varphi(n) = n\prod_{p|n}(1 - p^{-1}).$$

Let F be the set of integers $n > 1$ with the property that $\varphi(m) > \varphi(n)$ whenever $m > n$. The purpose of this paper is to establish a number of results about the set F. For example, we shall prove that each prime divides all sufficiently large elements of F, each positive integer divides some element of F, and that the ratio of successive elements of F approaches 1.

1. Introduction. Similar studies have been carried out in the past, initially by Ramanujan [7] for the divisor function $d(n)$, and then by Alaoglu and Erdös [1] for $d(n)$ and the divisor sum function $\sigma(n)$, and by Erdös and Nicolas [2] for the prime divisor function $\omega(n) = \Sigma_{p|n}1$ (see also the last paper for additional references). In particular, Ramanujan considered the set of integers n such that $d(m) < d(n)$ whenever $1 < m < n$. He called such integers highly composite, and by analogy it seems appropriate to refer to the elements of our set F as sparsely totient numbers.

Since $\varphi(n) \to \infty$ as $n \to \infty$, it is obvious that F is infinite. Our first result shows how to construct many elements of F explicitly. Let $p_1 = 2$, $p_2 = 3, \ldots$ denote the primes in ascending order of magnitude.

THEOREM 1. *Suppose $k \geq 2$, $d \geq 1$, $l \geq 0$ and*
(a) $d < p_{k+1} - 1$
(b) $d(p_{k+1} - 1) < (d + 1)(p_k - 1)$.
Then $dp_1 \cdots p_{k-1}p_{k+1}$ is in F.

COROLLARY. *Let n, n' be consecutive elements of F. Then $n'/n \to 1$ as $n \to \infty$.*

For $n > 1$ denote by $P(n)$ the greatest prime factor of n and by $Q(n)$ the smallest prime not dividing n. Already Theorem 1 above provides some information about large values of $P(n)$ and $Q(n)$ for n in F, as well as showing that there are elements of F divisible by any given integer d. Also, the statement that each prime divides all sufficiently large elements of F is equivalent to $Q(n) \to \infty$ as $n \to \infty$ in F. We shall prove this in much more precise form in our next result.