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SCHRÖDINGER SEMIGROUPS ON THE SCALE OF SOBOLEV SPACES

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We consider the action of semigroups e^{-tH}, with $H = -\Delta + V$ on $L^2(R^\nu)$, on the scale of Sobolev spaces $\mathscr{H}^\alpha$. We show that while e^{-tH} maps $L^2 = \mathscr{H}^0$ to $\mathscr{H}^2$ under great generality, there exist bounded V so that, for all $\beta > 0$, $e^{-tH}[\mathscr{H}^\beta]$ is not contained in any $\mathscr{H}^\alpha$ with $\alpha > 2$.

1. Introduction. This note represents a modest contribution to the issue of smoothing properties of Schrödinger semigroups, e^{-tH}, $H = -\Delta + V$ on $L^2(R^\nu)$ [**12**]. It has been shown [**3, 8, 2, 11, 12**] under fairly great generality (i.e. assumptions on V) that e^{-tH} is smoothing on the scale of L^p spacses, i.e. e^{-tH} maps L^p into any L^q with $q \geq p$. Kon [**7**] asked the question of smoothing properties on the scale of Sobolev spaces $\mathscr{H}^\alpha$. Below we will exploit their L^q analogs, so we define them: $f \in L^q(R^\nu)$ is said to lie in L^q_α $(\alpha \geq 0)$ if there exists $g \in L^q(R^\nu)$ so that $\hat{g}(p) = (1 + |p|^2)^{\alpha/2}\hat{f}(p) \cdot L^2_\alpha \equiv \mathscr{H}^\alpha$. We will also require the spaces K_ν defined initially by Kato [**5**]: If $\nu = 1$,

$$K_\nu = \left\{ f \,\middle|\, \sup_x \left[\int_{x-1}^{x+1} |f(y)|\, dy \right] < \infty \right\},$$

otherwise

$$K_\nu = \left\{ f \,\middle|\, \lim_{\alpha \downarrow 0} \left[\sup_x \int_{|x-y| \leq \alpha} B_\nu(x-y) |f(y)|\, d^\nu y \right] = 0 \right\}$$

where $B_\nu(x) = |x|^{-(\nu-2)}$ if $\nu \geq 3$ and $B_2(x) = -\ln|x|$. For any of these spaces χ, we define $\chi_{\text{loc}} = \{ f \mid f\varphi \in \chi \text{ for all } \varphi \in C_0^\infty(R^\nu)\}$. We summarize properties of these spaces needed below in an appendix.

Consider for a moment $\nu = 3$. It is well known [**6**] that if $V \in (L^2 + L^\infty)(R^3)$, then $D(H) = \mathscr{H}^2$, and thus obviously e^{-tH} maps $\mathscr{H}^0 = L^2$ to $\mathscr{H}^2$. Since there is lots of room between L^2 and L^∞, one might hope that for any $V \in L^\infty$, L^2 is mapped into some $\mathscr{H}^\alpha$ with $\alpha > 2$. Our main result in §2 will be to show there are $V \in L^\infty$ with compact support, so that $\operatorname{Ran}(e^{-tH})$ is not in any $\mathscr{H}^\alpha$ with $\alpha > 2$. Indeed, we will prove:

THEOREM 1. *Suppose that $V_+ = \max(V, 0) \in K_\nu^{\text{loc}}$ and $V_- = \max(-V, 0) \in K_\nu$ and that $He^{-tH}\varphi$ and $e^{-tH}\varphi$ lie in $\mathscr{H}^\alpha_{\text{loc}}$ for some $\alpha > 2$ and for one $\varphi \geq 0$ $(\varphi \not\equiv 0)$. Then for $\beta = \min(\alpha - 2, 1)$, $V \in L^{4/3}_{\beta,\text{loc}}$.*