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UNIMODULAR APPROXIMATION IN FUNCTION ALGEBRAS

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Let A be a function algebra on the compact Hausdorff space X. The main result of this paper gives necessary and sufficient conditions for the set of quotients of inner functions in A to be dense in the set of continuous unimodular functions on X. A theorem of Douglas and Rudin concerning quotients of Blaschke products is derived. The main result is also applied in the context of the theory of compact abelian groups.

The prototype for the main result of this paper is the following theorem due to Douglas and Rudin [3].

THEOREM 1. *Suppose that u is a measurable function on the unit circle such that $|u(e^{it})| = 1$ a.e. Then u can be approximated arbitrarily closely in the essential-sup norm by quotients of Blaschke products.*

We will derive Theorem 1 from a general result concerning function algebras. Our main result will also be applied to certain algebras of functions on compact abelian groups.

Preliminaries. Consider a sub-algebra A of the algebra $C(X)$ of continuous complex-valued functions on the compact Hausdorff space X. We will assume that A contains the constant functions and is closed with respect to the sup-norm $\|\cdots\|$. We will call a function $g \in A$ *inner* if $|g| \equiv 1$. Let $I(A)$ denote the set of inner functions and let $Q(A)$ denote the set of quotients of inner functions. Of course the set $U(X) = I(C(X))$ is simply the collection of continuous unimodular functions on X. $U(X)$ is a group under pointwise multiplication. $U(X)$ has an important subgroup, namely the group $\log U(X)$ of members of $U(X)$ having continuous logarithms. We will indicate the natural quotient map from $U(X)$ to $U(X)/\log U(X)$ by π_X.

In the case where X is the n-dimensional torus T^n and A is the polydisk algebra $A(T^n)$, i.e., the closed algebra generated by the coordinate projections from T^n onto T^1, there is an abundance of inner functions. It is known that a function $f \in A(T^n)$ is inner if and only if it is of the form

$$f(z) = M(z)\hat{Q}(z)/Q(z), \qquad z = (z_1, z_2, \ldots, z_n),$$