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A NOTE ON ORDERINGS ON ALGEBRAIC VARIETIES

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It was proven in [A–G–R] that if $V \subset \mathbf{R}^n$ is a surface and α a total ordering in its coordinate polynomial ring, α can be described by a half branch (i.e., there exists $\gamma(0,\varepsilon) \to V$, analytic, such that for every $f \in \mathbf{R}[V]\, \mathrm{sgn}_\alpha f = \mathrm{sgn} f(\gamma(t))$ for t small enough). Here we prove (in any dimension) that the orderings with maximum rank valuation can be described in this way. Furthermore, if the ordering is centered at a regular point we show that the curve can be extended C^∞ to $t = 0$.

1. (1.0) Let V be an algebraic variety over $\mathbf{R}$ and α an ordering in $K = \mathbf{R}(V)$. If α is described by a half-branch γ: $(0,\varepsilon) \to V$, no non-zero polynomial vanishes over $\gamma(t)$ for t small enough. Consequently, if V' is birrationally equivalent to V (i.e., $\mathbf{R}(V') = \mathbf{R}(V)$), $\alpha \cap \mathbf{R}[V']$ is also described by a curve in V'.

(1.1) PROPOSITION. *Let V be an algebraic variety over $\mathbf{R}$ and $n = \dim V$. If $\mathbf{R}[V]$ is an integral extension of $\mathbf{R}[x_1,\dots,x_n] = \mathbf{R}[\underline{x}]$ and α an ordering on $\mathbf{R}[V]$ such that $\beta = \alpha \cap \mathbf{R}[\underline{x}]$ can be described by a half-branch, then the same holds true for α.*

Proof. By our previous remark (1.0) we can suppose V is a hypersurface. Thus $\mathbf{R}[V] = \mathbf{R}[\underline{x}, x_{n+1}](P)$ where $P \in \mathbf{R}[\underline{x}][x_{n+1}]$ is a monic polynomial in x_{n+1}. Let δ be the discriminant of P and π: $V \to \mathbf{R}^n$ the projection on the first n-coordinates. Then the restriction

$$\pi_|: V \setminus \pi^{-1}(\delta = 0) \to \mathbf{R}^n \setminus \{\delta = 0\}$$

has finite fibers with constant cardinal over every connected component. Moreover, by the implicit function theorem, $\pi_|$ is an analytic diffeomorphism from every connected component of $V \setminus \pi^{-1}(\delta = 0)$ onto someone of $\mathbf{R}^n \setminus \{\delta = 0\}$.

Let γ: $(0,\varepsilon) \to \mathbf{R}^n$ be the curve describing β. The connected components $C_1,\dots,C_p$ of $\mathbf{R}^n \setminus \{\delta = 0\}$ are open semi-algebraic sets, and we can write

$$C_i = \bigcup_{j=1}^{q} \{f_{ij1} > 0,\dots,f_{ijr} > 0\}, \qquad f_{ijl} \in \mathbf{R}[\underline{x}].$$