PACIFIC JOURNAL OF MATHEMATICS
Vol. 126, No. 2, 1987

A CHARACTERIZATION OF KK-THEORY

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We characterize the KK-groups of G. G. Kasparov, along with the Kasparov product $KK(A,B) \times KK(B,C) \to KK(A,C)$, from the point of view of category theory (in a very elementary sense): the product is regarded as a law of composition in a category and we show that this category is the universal one with "homotopy invariance", "stability" and "split exactness". The third property is a weakened type of half-exactness: it amounts to the fact that the KK-groups transform split exact sequences of C^*-algebras to split exact sequences of abelian groups. The method is borrowed from Joachim Cuntz's approach to KK-theory, in which cycles for $KK(A,B)$ are regarded as generalized homomorphisms from A to B: the results follow from an analysis of the Kasparov product in this light.

Introduction. This paper is a study of the groups $KK(A,B)$, where A and B are separable C^*-algebras, introduced by G. G. Kasparov in [**15**]. These groups have received widespread attention since their introduction, due mainly to the possibilities they afford for the application of C^*-algebra techniques to problems in geometry and topology, but also because of their utility within the field of C^*-algebras. The groups arise from, and generalize, the topological K-theory of spaces—thus if X is a compact metric space then the groups $KK(\mathbf{C}, C(X))$ and $KK(C(X), \mathbf{C})$ are respectively the topological K-theory and K-homology of X—and as such their introduction has led to, for example, simplifications and a conceptualization of the proof of the Atiyah Singer Index Theorem (see [**4**], [**13**], [**9**]). More importantly, by using non-commutative C^*-algebras as arguments for the KK-groups the index theorem can be generalized in a number of interesting directions (for example, to foliations [**9**]). As another example of an application of these groups in topology, the group $KK(C^*(G), \mathbf{C})$ serves as an appproximation to the group $K(BG)$ (where G is say the fundamental group of a manifold) and a study of it as such has led to progress in the generalized Novikov conjecture (see e.g. [**18**] for a discussion of this). As a tool in the study of C^*-algebras, they are of importance as a relatively computable invariant, as well as in the study of extensions of C^*-algebras developed in [**6**] and [**15**].

The definition of the KK-groups originates in the close relationship between K-theory and the index theory of elliptic operators. An elliptic differential (or pseudodifferential) operator on a smooth closed manifold