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THE BEHAVIOR OF CHAINS OF ORDERINGS UNDER FIELD EXTENSIONS AND PLACES

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In the Artin-Schreier theory of formally real fields, the isomorphism classes of certain extensions of a formally real field (namely, the real closures) are shown to correspond bijectively with certain arithmetic structures (namely, orderings) which these extensions induce on the base field. E. Becker has generalized the notion of a real closure (by "real closures at orderings of higher level"), and the isomorphism classes of these generalized real closures again correspond bijectively to certain arithmetic invariants they induce on the base field; these are in essence Harman's "chains of orderings". This paper includes a rather complete analysis of the behavior of such chains of orderings up and down both field extensions and places. The analysis of this behavior is reduced to tractable problems in abelian group theory (together with the analysis of the behavior of ordinary orderings under extensions and places).

1. Introduction. Recall that an *ordering of higher level* (abbreviated: "ordering") of a field F is a subset of F maximal with respect to exclusion of -1 and closure under multiplication and addition (i.e., a Harrison prime) which contains F^{2^n} for some $n \geq 1$ (the least such n is the *exact level* of the ordering) [**B**]. Following Lam [**L**], we will call the orderings of exact level one (i.e., those which figure in the Artin-Schreier theory) *ordinary orderings*. A *chain of orderings* of a field F [**H**] is a sequence $(P_i)_{i \geq 0}$ such that P_0 is an ordinary ordering of F and for each $i > 0$, P_i is an ordering of F of exact level i such that $ZP_i = Z(P_{i-1} \cap P_0)$. (For any $A \subset F$, we are denoting by ZA the set of integer multiples of elements of A. Note that if $(P_i)_{i \geq 0}$ is a chain of orderings in the above sense, then $P_0 \neq P_1$ since otherwise $ZP_2 = ZP_0 = F$, contradicting that P_2 has level 2. One now checks easily that the above definition is equivalent to Harman's original definition.)

Now let us fix an ordinary ordering P of a field F. We wish to calculate $\mathbf{C}(P)$, the set of all sequences $(P_i)_{i \geq 0}$ which are either chains of orderings with $P_0 = P$, or which have $P_i = P$ for all $i \geq 0$. Let Γ denote the value group of the real-valued place induced by P (i.e., Γ is the group of Archimedean classes). In §2, we will give a natural bijection

$$\Psi_P \colon \mathbf{C}(P) \to \operatorname{Hom}(\Gamma, I_2)/\sim$$