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SOME CHARACTERIZATIONS OF ANALYTIC METRIC SPACES

SOPHOCLES MERCOURAKIS

For a topological space X, let $C_1(X)$ denote the Banach space of all bounded functions $f\colon X \to \mathbf{R}$ such that for every $\varepsilon > 0$ the set $\{x \in X\colon |f(x)| \geq \varepsilon\}$ is closed and discrete in X, endowed with the supremum norm. Using spaces of this form we give a direct proof (Corollary 1.5) of a result of Dashiell and Lindenstrauss on strict convexity of Banach spaces. Subsequently we obtain two types of characterizations of analytic metric spaces. The first (Theorem 2.3) is topological and is based on the set theoretic ordering of the compact subsets of X; this is related to some results of Christensen and Talagrand. The second (Theorem 3.1) is functional analytic and is based on the existence of bounded linear operators between the spaces of the form $C_1(X)$.

1. A simple example of Dashiell-Lindenstrauss type. In this section we introduce the class of Banach spaces $C_1(X)$, where X is a topological space and give a simple proof that there is no bounded linear one-to-one operator from $C_1(\Sigma)$, where Σ is the Baire space ω^ω (of infinite sequences of natural numbers with the product topology) into the classical Banach space $C_0(\Gamma) = \{f\colon \Gamma \to \mathbf{R}\colon$ for every $\varepsilon > 0$ the set $\{\gamma \in \Gamma\colon |f(\gamma)| \geq \varepsilon\}$ is finite$\}$, for any set Γ. Since $C_1(\Sigma)$ is strictly convexifiable [**7, 8**] the Banach space $C_1(\Sigma)$ is still another example with the properties of the examples of Dashiell-Lindenstrauss (see [**2**], Theorem 2).

DEFINITION 1.1. Let X be a topological space. We set $C_1(X) = \{f\colon X \to \mathbf{R}\colon f$ is bounded and for every $\varepsilon > 0$ the set $\{t \in X\colon |f(t)| \geq \varepsilon\}$ is closed and discrete in $X\}$. It is clear that $C_1(X)$ with supremum norm is a Banach space.

We notice that:

(a) For a subset $A \subseteq X$, A is closed and discrete in X if and only if A', the derived set of A, is empty;

(b) $C_0(X) \subseteq C_1(X)$, and if $f \in C_1(X)$ then $f|\Omega \in C_0(X)$ for all compact subsets Ω of X;

(c) if A is compact (resp. closed and discrete) in X, then $C_0(A)$ (resp. $l^\infty(A)$ is a complemented subspace of $C_1(X)$). In particular, if X is compact (resp. discrete) then $C_1(X) = C_0(X)$ (resp. $C_1(X) = l^\infty(X)$);