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CHARACTERIZATION OF C^*-ALGEBRAS WITH CONTINUOUS TRACE BY PROPERTIES OF THEIR PURE STATES

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We characterize C^*-algebras with continuous trace among all C^*-algebras by a condition on the set $P(A)$ of pure states. The condition is that (1) the graph $R(A)$ of the unitary equivalence relation on $P(A)$ is closed in $P(A) \times P(A)$, and (2) transition probabilities are continuous for the product topology on $R(A)$ (i.e. that inherited from $P(A) \times P(A)$). If $R(A)$ is given the quotient topology, these conditions are equivalent to properness of the inclusion map from $R(A)$ into $P(A) \times P(A)$. We show the product and quotient topologies on $R(A)$ coincide iff transition probabilities are continuous for the product topology, and this in turn is equivalent to Fell's condition. Transition probabilities are always continuous for the quotient topology on $R(A)$.

Introduction. In **[15]** it was shown that the set $P(A)$ of pure states of a C^*-algebra A determines A up to $*$-isomorphism. Here $P(A)$ carries the structure of a uniform space (for the weak* uniformity), transition probabilities (or equivalently, the distance given by the norm on $P(A) \subseteq A^*$), and orientation.

In §2 of the current paper we explore a connection between two of these structures by studying weak* continuity of transition probabilities. Continuity on all of $P(A) \times P(A)$ is rare: it occurs only for A equal to a c_0 direct sum of elementary C^*-algebras. (Equivalently: A is type I and $\hat{A}$ is discrete.) The set $R(A)$ of pairs of unitarily equivalent pure states provides a more interesting domain for transition probabilities. For the topology inherited as a subspace of $P(A) \times P(A)$ (which we call the product topology on $R(A)$), continuity of transition probabilities restricted to $R(A)$ is equivalent to Fell's condition. (Roughly following the terminology in **[4]**, we say a C^*-algebra A satisfies Fell's condition if for each π_0 in $\hat{A}$ there is an element b in A^+ such that $\pi(b)$ is a projection of rank one for all π in some neighborhood of π_0.) This gives the characterization of continuous trace algebras described in the abstract above.

There is a second topology on $R(A)$ that is of great interest. Let $G(A)$ be the set of extreme points of the unit ball A^*. If ϕ is in $G(A)$,