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COORDINATES FOR TRIANGULAR OPERATOR ALGEBRAS II

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Dedicated to the memory of Henry A. Dye

Let M be a von Neumann algebra and let A be a maximal abelian self-adjoint subalgebra (masa) of M. A subalgebra $\mathfrak{T}$ of M is called *triangular* (with respect to A) if $\mathfrak{T} \cap \mathfrak{T}^* = A$, where $\mathfrak{T}^*$ denotes the collection of adjoints of the elements in $\mathfrak{T}$. If $\mathfrak{T}$ is not contained in any larger triangular subalgebra of M, then $\mathfrak{T}$ is called maximal triangular. If A is a Cartan subalgebra, then M may be realized as an algebra of matrices indexed by an equivalence relation on a standard Borel space and if $\mathfrak{T}$ is σ-weakly closed and maximal triangular, then $\mathfrak{T}$ may be realized as the collection of matrices supported on the graph of a partial order that totally orders each equivalence class. In this paper we will be concerned with the relation between the structure of these algebras and the theory of analytic operator algebras. It turns out that this relation is complex: it involves the cohomology of the equivalence relation, the order type of the partial order and the type of M.

The concept of a triangular algebra was introduced formally in 1959 by Kadison and Singer in their fundamental paper **[KS]** which launched the theory of non-self-adjoint operator algebras. Their objective was to develop a theory of operator algebras whose elements could simultaneously all be put into triangular form. It is an elementary exercise to see that when M is the full algebra of n by n matrices and when A is the subalgebra of diagonal matrices, a masa in M, then given a triangular subalgebra $\mathfrak{T}$ of M, it is possible to conjugate $\mathfrak{T}$ by a permutation matrix so that the matrices in $\mathfrak{T}$ are all upper triangular. Moreover, $\mathfrak{T}$ is maximal triangular if and only if $\mathfrak{T}$ is (unitarily equivalent) to the algebra of all upper triangular matrices. In **[MSS]** we began a study of triangular algebras where it is assumed that A is a Cartan subalgebra of M in the sense of Feldman and Moore **[FM1]**. (This means that there is a faithful normal expectation from M onto A and that the group of unitary operators in M that normalizes A generates M.) Feldman and Moore showed that if A is a Cartan subalgebra of M, then there is a standard Borel measure space $(X, \mathscr{B}, \mu)$ and a Borel equivalence relation $R \subseteq X \times X$, with the property that