PACIFIC JOURNAL OF MATHEMATICS
Vol. 139, No. 1, 1989

SPACES OF WHITNEY MAPS

ALEJANDRO ILLANES

Let X be a continuum. Let 2^X (respectively, $C(X)$) be the hyperspace of nonempty closed subsets (respectively, subcontinua) of X, endowed with the Hausdorff metric. For $\mathscr{H} = C(X)$ or 2^X, let $W(\mathscr{H})$ denote the space of Whitney maps for $\mathscr{H}$ with the "sup metric" and pointwise product. In this paper we prove that if there exists a homeomorphism ϕ: $W(C(X)) \to W(C(Y))$ (or ϕ: $W(2^X) \to W(2^Y)$) which preserved products and "strict order", then X is homeomorphic to Y. We also prove that there exists an embedding ψ: $W(C(X)) \to W(2^X)$ such that $\psi(u)$ is an extension of u for each $u \in W(C(X))$.

Introduction. A *continuum* is a nondegenerate compact, connected metric space. All the spaces considered here are continua. If X is a continuum, 2^X (respectively, $C(X)$) is the hyperspace of nonempty closed subsets (respectively, subcontinua) of X, endowed with the Hausdorff metric H. Let $\mathscr{H}$ be a nonempty closed subset of 2^X. A *Whitney map* for $\mathscr{H}$ is a continuous function u: $\mathscr{H} \to [0,1]$ such that (a) $u(A) = 0$ if and only if A is a single point set; (b) $u(A) = 1$ if and only if $A = X$; and (c) if $A, B \in \mathscr{H}$ and $A \subset B \neq A$, then $u(A) < u(B)$. Let $W(\mathscr{H})$ denote the space of Whitney maps for $\mathscr{H}$. We identify X with $\{\{x\}: x \in X\} \subset C(X), 2^X$. Given $u, w \in W(\mathscr{H})$, we say that u is *strictly smaller* than w ($u \lhd w$) if $u(A) < w(A)$ for each $A \in \mathscr{H} - (X \cup \{X\})$ and u is *smaller or equal* than w ($u \leq w$) if $u(A) \leq w(A)$ for each $A \in \mathscr{H}$. We consider $W(\mathscr{H})$ with the "sup metric", the pointwise product and the orders defined above.

In this paper we prove that: (a) $W(\mathscr{H})$ is a topologically complete space. This answers a question asked by S. B. Nadler, Jr. [**2**, question 14.71.4]; (b) There is a natural way to embed 2^X in $W(C(X))$ and in $W(2^X)$; (c) If $\mathscr{H} = C(X)$ and $\mathscr{G} = C(Y)$ or $\mathscr{H} = 2^X$ and $\mathscr{G} = 2^Y$ and there exists a homeomorphism ϕ: $W(\mathscr{H}) \to W(\mathscr{G})$ which is a semigroup isomorphism and preserves strict order (in the sense that $u \lhd w$ if and only if $\phi(u) \lhd \phi(w)$), then X is homeomorphic to Y (this answers, partially, question 14.71.1 formulated by S. B. Nadler, Jr. in [**2**]). In [**3**], L. E. Ward, Jr., showed that Whitney maps for $\mathscr{H}$ can be extended to Whitney maps for 2^X. Using his constructions, we