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SMALL SUBSET OF THE PLANE WHICH ALMOST CONTAINS ALMOST ALL BOREL FUNCTIONS*

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A Borel subset B of the plane is constructed which is small from the Lebesgue measure point of view and large in the sense of the Baire category. All vertical sections of B have measure zero, and for each Borel function $f: R \to R$ for all but countably many y the set $\{x \in R: (x, f(x)+y) \in B\}$ is comeager.

1. Introduction. It is well known that the Fubini Theorem and the Kuratowski-Ulam Theorem cannot be mixed together. It is easy to find a Borel subset of the plane with all vertical sections having the Lebesgue measure zero and almost all in the sense of the Baire category horizontal sections being comeager. We can just take A^2 where A is the classical example of a measure zero dense G_δ subset of the real line.

In this paper we show that such antagonism between measure and category is much stronger. We give an example of a G_δ subset of the plane such that all its vertical sections have the Lebesgue measure zero and for each Borel function $f: R \to R$ all but countably many sections parallel to f are comeager. We also indicate why there is no such example in which the roles of measure and category are interchanged.

2. Notation. Throughout the paper I stands for the half open interval $[0, 1[$, R for the reals, $+_a$ for the addition modulo a, and λ for the Lebesgue measure. For subsets A and B of the reals let $A \mp B = \{a \mp b: a \in A \text{ and } b \in B\}$. Natural numbers are the sets of smaller natural numbers, ω is the set of all natural numbers, $\omega^{<\omega}$ and ω^ω are the sets of finite, resp. infinite, sequences of natural numbers. $[A]^{<\varkappa}$ and $[A]^\varkappa$ denote the sets of all subsets of A of cardinality $< \varkappa$, resp. of cardinality $\varkappa$. For $A \subseteq X \times Y$ and $x \in X$ let $A_x = \{y \in Y: (x, y) \in A\}$.

3. Erdös-Folklore Lemma. We start with the following Folklore Lemma.

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