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NEVANLINNA PARAMETRIZATIONS FOR THE EXTENDED INTERPOLATION PROBLEM

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Let $\mathscr{B}$ be the set of holomorphic functions f with $|f| \leq 1$ in the open unit disc $D = \{z \in \mathbb{C}: |z| < 1\}$. Let $\sigma = \{z_1, z_2, \dots\}$ be a finite or infinite sequence of distinct points in D and, for each point $z_i \in \sigma$, $(c_{i0}, \dots, c_{in_i-1})$ be an ordered n_i-tuple of complex numbers $(0 < n_i < +\infty)$. The problem is to find a function f which belongs to $\mathscr{B}$ and satisfies the extended interpolation conditions

$$\text{(EI)} \quad f(z) = \sum_{\alpha=0}^{n_i-1} c_{i\alpha}(z-z_i)^\alpha + \mathbf{O}((z-z_i)^{n_i}) \qquad (\forall z_i \in \sigma).$$

Let $\mathscr{E}$ denote the set of all solutions of this problem (EI) in $\mathscr{B}$ and assume the hypothesis

$$\text{(H)} \qquad \mathscr{E} \text{ has at least two elements.}$$

A bijection $\pi: \mathscr{B} \to \mathscr{E}$ is called *Nevanlinna parametrization of* $\mathscr{E}$ if there exist four functions P, Q, R, and S holomorphic in D and such that $Rg + S \not\equiv 0$, $\pi(g) = (Pg+Q)/(Rg+S)$ for any $g \in \mathscr{B}$. The existence, some properties and some applications of such parametrizations are shown. One has a bijection between the set of Nevanlinna parametrizations of $\mathscr{E}$ and the group of Möbius transformations.

In our previous paper **[24]**, $\sigma = \{z_1, \dots, z_k\}$ being finite, we constructed in a simple manner an Hermitian $n \times n$ matrix A from the given data $\{z_i\}$ and $\{c_{i\alpha}\}$ $(1 \leq i \leq k, 0 \leq \alpha \leq n_i - 1)$, where $n = \sum_{i=1}^k n_i$, and established the following theorems which unify the coefficient theorem of Carathéodory-Toeplitz-Schur **[3]**, **[26]**, and **[19]** and the interpolation theorem of Pick **[17]**:

THEOREM E. *The problem* (EI) *admits a solution in $\mathscr{B}$ if and only if $A \geq 0$ (positive semidefinite). If so, among the solutions, there is a finite Blaschke product of degree $\leq n$.*

THEOREM U. *The solution of the problem* (EI) *in $\mathscr{B}$ is unique if and only if $A \geq 0$ and $\det A = 0$. In this case, the unique solution is a finite Blaschke product whose degree is equal to the rank of A.*