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ON THE ELIMINATION OF ALGEBRAIC INEQUALITIES

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Let S be a locally closed semi-algebraic subset of $\mathbb{R}^n$. We find an irreducible equation of an algebraic set of $\mathbb{R}^{n+1}$ projecting upon S. Our methods are simple and explicit.

1. Introduction. The inequality $x \geq 0$ is often replaced by the proposition "x has a square root" or "$\exists t \in \mathbb{R},\ t^2 - x = 0$". This is the most immediate example of an elimination of one inequality. The general problem is to find an algebraic set projecting upon a given semi-algebraic set: it is a converse of the problem of the elimination of quantifiers.

Motzkin proved that every semi-algebraic subset of $\mathbb{R}^n$ is the projection of an algebraic set in $\mathbb{R}^{n+1}$. However this algebraic set is very complicated and generally reducible.

Andradas and Gamboa proved that any closed semi-algebraic subset of $\mathbb{R}^n$ whose Zariski-closure is irreducible is the projection of an irreducible algebraic set in $\mathbb{R}^{n+k}$.

In this paper we shall first improve Motzkin's result by finding equations generally of minimal degree. Then we shall give a few results concerning irreducibility. One of the first examples of such a construction is due to Rohn and has been studied by Hilbert and Utkin: If $4C_4C_2 = \varepsilon^2$ is a plane curve of degree six (where $\deg(C_2) = 2$, $\deg(C_4) = 4$, $\varepsilon \in \mathbb{R}$), then it is the apparent contour of the quartic surface $C_2z^2 - \varepsilon z + C_4 = 0$.

2. The case of basic closed subsets. Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$ be the set of nonnegative numbers. Let $\mathbf{x} = (x_1, \dots, x_N)$ be a "parameter" and t an "indeterminate", so that we can speak of the roots of a polynomial $P(\mathbf{x}, t)$. In the same way, unless otherwise specified, the degree of $P(\mathbf{x}, t)$ will be its degree in t.

Let us define the polynomials $a_i(\mathbf{x})$ as follows:

$$a_k(x_1, \dots, x_{k+1}) = x_{k+1}(x_1 + x_2 + \cdots + x_k).$$

It is easy to see that $a_1(\mathbf{x}) \geq 0, \dots, a_n(\mathbf{x}) \geq 0$ if and only if all the x_i are nonnegative or all the x_i are nonpositive $(i = 1, \dots, n+1)$.