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A COMPARISON ALGEBRA ON A CYLINDER WITH SEMI-PERIODIC MULTIPLICATIONS

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A necessary and sufficient Fredholm criterion is found for a C^*-algebra of bounded operators on a cylinder, which contains operators of the form $L\Lambda^M$, where $\Lambda = (1-\Delta)^{-1/2}$ and L is an Mth order differential operator whose coefficients are periodic at infinity.

0. Introduction. Let Ω denote the cylinder $\mathbb{R}\times\mathbb{B}$, where $\mathbb{B}$ is a compact Riemannian manifold, Δ_Ω its Laplacian and $\mathscr{H}$ the Hilbert space $L^2(\Omega)$. Cordes [3] found a necessary and sufficient Fredholm criterion for operators in the C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ generated by: (i) multiplications by functions that extend continuously to $[-\infty, +\infty]\times\mathbb{B}$, (ii) $\Lambda = (1-\Delta_\Omega)^{-1/2}$ and (iii) operators of the form $D\Lambda$, where D is either $\partial/\partial t$, $t\in\mathbb{R}$, or a first order differential operator on $\mathbb{B}$ with smooth coefficients. Here we extend this algebra by adjoining the multiplications by 2π-periodic continuous functions to the generators, and a similar Fredholm criterion is obtained.

The commutator ideal $\mathscr{E}_{\mathscr{P}}$ of the extended algebra $\mathscr{C}_{\mathscr{P}}$ is proven to be $*$-isomorphic to $\mathscr{SL}\overline{\otimes}\mathscr{K}_{\mathbb{Z}}\overline{\otimes}\mathscr{K}_{\mathbb{B}}$, where $\mathscr{SL}$ denotes the algebra of singular integral operators on the circle and $\mathscr{K}_{\mathbb{Z}}$ and $\mathscr{K}_{\mathbb{B}}$ denote the algebras of compact operators on $L^2(\mathbb{Z})$ and $L^2(\mathbb{B})$, respectively. This allows us to define on $\mathscr{C}_{\mathscr{P}}$ an operator-valued symbol, the "γ-symbol", such that $\ker\gamma\cap\ker\sigma$ equals the compact ideal of $\mathscr{L}(\mathscr{H})$. Here σ denotes the complex-valued symbol on $\mathscr{C}_{\mathscr{P}}$ that arises from the Gelfand map of the commutative C^*-algebra $\mathscr{C}_{\mathscr{P}}/\mathscr{E}_{\mathscr{P}}$. We prove that $A\in\mathscr{C}_{\mathscr{P}}$ is Fredholm if and only if γ_A and σ_A are invertible.

The simpler case when the compact manifold reduces to a point is considered in [5]. There, a unitary map W from $L^2(\mathbb{R})$ onto $L^2(S^1)\overline{\otimes}L^2(\mathbb{Z})$ is defined, such that the conjugate $W\mathscr{E}W^{-1}$ of the commutator ideal equals $\mathscr{SL}\overline{\otimes}\mathscr{K}_{\mathbb{Z}}$. Here, we conjugate $\mathscr{E}_{\mathscr{P}}$ with $W\otimes I_{\mathbb{B}}$, where $I_{\mathbb{B}}$ denotes the identity operator on $L^2(\mathbb{B})$, and obtain $\mathscr{SL}\overline{\otimes}\mathscr{K}_{\mathbb{Z}}\overline{\otimes}\mathscr{K}_{\mathbb{B}}$.

If L is a differential operator on Ω whose coefficients are continuous and approach periodic functions at infinity, the operator $A = L\Lambda^M$ belongs to $\mathscr{C}_{\mathscr{P}}$, where M is the order of L. We can apply