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A DUALITY THEOREM FOR EXTENSIONS OF INDUCED HIGHEST WEIGHT MODULES

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We begin by recalling that homogeneous differential operators between smooth vector bundles over a real flag manifold correspond to the intertwining maps between algebraically induced highest weight modules. Within this framework we prove a duality theorem for extensions of induced highest weight modules. In particular, this leads to a duality theory for the nilpotent cohomology of any generalized Verma module.

1. Introduction. In this short note we recall (and prove) a folklore result which appeared in Boe's 1982 Yale thesis (without proof) and was attributed to G. Zuckerman. We apply the result to representations of real groups and to the theory of highest weight modules. In particular we obtain a duality theorem for extensions between parabolically induced highest weight modules, cf. Theorem 1.1 below. Nontrivial applications are discussed in §§4 and 5.

Fix a pair $(\mathfrak{g}, \mathfrak{p})$, $\mathfrak{g}$ a complex semisimple Lie algebra and $\mathfrak{p}$ a parabolic subalgebra. There exists a connected real semisimple matrix group G with a closed parabolic subgroup P so that $\mathfrak{g}$ and $\mathfrak{p}$ are the complexified Lie algebras of G and P respectively. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be a Levi decomposition of $\mathfrak{p}$ and $\mathfrak{h} \subseteq \mathfrak{l}$ a Cartan subalgebra of both $\mathfrak{l}$, the reductive part of $\mathfrak{p}$, and of $\mathfrak{g}$.

Recall the category $\mathscr{O}_\mathfrak{p}$ of finitely generated $\mathfrak{g}$-modules which are $\mathfrak{l}$-semisimple and $\mathfrak{p}$-locally finite. Denote by $\mathscr{A}_\mathfrak{p}$ the category of finite dimensional $\mathfrak{p}$-modules which are $\mathfrak{l}$-semisimple. Define a functor $U_\mathfrak{p}: \mathscr{A}_\mathfrak{p} \to \mathscr{O}_\mathfrak{p}$ by

$$U_\mathfrak{p}(E) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E.$$

Here $U(\mathfrak{a})$ denotes the enveloping algebra of a Lie algebra $\mathfrak{a}$. For any finite dimensional $\mathfrak{p}$-module (or P-module) E, let E^* denote the contragradient module. Our main result is then

THEOREM 1.1. *For any two $\mathfrak{p}$-modules E and F in $\mathscr{A}_\mathfrak{p}$ and any $k \geq 0$,*

$$\dim_\mathbb{C} \operatorname{Ext}^k_{\mathscr{O}_\mathfrak{p}}(U_\mathfrak{p}(E), U_\mathfrak{p}(F)) = \dim_\mathbb{C} \operatorname{Ext}^k_{\mathscr{O}_\mathfrak{p}}(U_\mathfrak{p}(F^*), U_\mathfrak{p}(E^*)).$$