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FOURIER COEFFICIENTS OF NON-HOLOMORPHIC MODULAR FORMS AND SUMS OF KLOOSTERMAN SUMS

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This paper studies Fourier coefficients of non-holomorphic modular forms and sums of Kloosterman sums.

1. Introduction. Put $\Gamma = \mathrm{PSL}(2, Z)$ and $H^+ = \{x + iy | y > 0\}$. Consider the Hilbert space $\mathfrak{L}^2(H^+/\Gamma)$ of function $u(z)$ satisfying:

$$u(\gamma z) = u(z) \qquad (\gamma \in \Gamma)$$

and

$$\langle u, u \rangle = \iint_{H^+/\Gamma} |u(z)|^2 \frac{dx\,dy}{y^2} < +\infty .$$

Consider the Laplacian Δ on $\mathfrak{L}^2(H^+/\Gamma)$:

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) .$$

A function $u(z)$ in $\mathfrak{L}^2(H^+/\Gamma)$ is called a cusp form if the constant term in the Fourier expansion of $u(z)$ vanishes. It is known that the Laplacian Δ has a complete discrete spectral decomposition on the subspace of cusp forms. The Maass wave forms $u_j(z)$ defined by

$$(1) \qquad \Delta u_j(z) = \lambda_j u_j(z), \quad \langle u_j, u_j \rangle = 1,$$

where $\lambda_1 \le \lambda_1 \le \lambda_3 \le \cdots$ are the discrete eigenvalues of Δ, constitute an orthonormal basis for the subspace of cusp forms. Note that $\lambda_1 > \frac{3}{2}\pi^2$. From (1) we have the Fourier expansion:

$$(2) \qquad u_j(z) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{ik_j}(2\pi |n) e(nx), \quad e(\theta) = e^{2\pi i \theta}$$

where $\lambda_j = \frac{1}{4} + k_j^2$ and $K_{ik_j}(\cdot)$ is the Whittaker function. We have

$$(3) \qquad \#\{k_j | \, |k_j| \le X\} = \tfrac{1}{12} X^2 + cX \log X + O(X)$$

where c is a constant; cf. Venkov [7].