PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 2, 1991

SETS OF UNIQUENESS AND SYSTEMS OF INEQUALITIES HAVING A UNIQUE SOLUTION

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Suppose a number of X-ray pictures is taken of the same object, but from different directions. One typically likes to know to what degree the pictures determine the object and exactly when an object is uniquely determined. Replacing picture taking by projections, that is, images relative to specified mappings, these same problems are easily formulated for higher dimensions and even for abstract spaces. The objects on hand might be data structures.

With this general framework, starting from an arbitrary but fixed collection of mappings, we study a new and very useful class of objects (sets) each of which is uniquely determined by its projections. In the process, we disprove a previously conjectured characterization of uniqueness relative to the one-dimensional projections in R^n. For all situations where the underlying space is finite, a complete and rather simple characterization of uniqueness is obtained.

1. Introduction. Suppose an X-ray picture is made of an object S in R^3 of uniform unit density. This corresponds to the creation of an image of S on a photographic plate Y by means of a central or parallel projection $\pi: R^3 \to Y$. The darkness of the image at $y \in Y$ is directly related to the length $|L \cap S|$ of the part of L inside S, with L as the straight line $L = \pi^{-1}\{y\} = \{x \in R^3: \pi x = y\}$. Hence, having such a π-photograph of S is equivalent to knowing the precise value $\lambda(S \cap A)$, for each set of the form $A = \pi^{-1}B = \{x \in R^3: \pi x \in B\}$. Here, B can be any subset of Y while λ is Lebesgue measure on R^3.

Given any finite class of such photographic maps $\pi_j: R^3 \to Y_j$, we would like to know exactly what subsets S of R^3 are such that S is uniquely determined by its set of images.

The subset S of R^3 can be identified with the measure μ_S on R^3 defined by $\mu_S(A) = \lambda(S \cap A)$. Thus μ_S has its density relative to Lebesgue measure λ equal to the function $1_S(x)$ on X (1 on S and 0 on its complement S^c). If $\pi: R^3 \to Y$ is *any* map then the π-projection of μ_S onto Y (also nonchalantly called the π-projection of S) is the mass distribution (measure) $\pi\mu_S$ on Y, whose mass inside any subset of B of Y equals $\mu_S(A) = \lambda(S \cap A)$ with $A = \pi^{-1}B = \{x \in R^3: \pi x \in B\}$. Knowing the π-photograph of S is the same as knowing the projection $\pi\mu_S$ of μ_S.