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ROBIN FUNCTIONS AND ENERGY FUNCTIONALS OF MULTIPLY CONNECTED DOMAINS

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The Robin function of a planar domain is a generalization of Green's function. It can be used to represent the solutions of mixed boundary-value problems for harmonic functions. Here it is combined with a variational method to solve certain extremal problems for the energy functional of a multiply connected domain. Some deeper properties of the Robin function are then explored. An allied system of conformal invariants called the Robin matrix is introduced and is compared with the classical Riemann matrix of a finitely connected domain.

In potential theory one considers three important boundary-value problems: to determine a harmonic function u in a given domain from its values on the boundary, from the values of its normal derivative on the boundary, and from the values of $\partial u/\partial n + h(s)u$ on the boundary, where $h(s)$ is a given positive function of the arclength. The problems are solved by means of Green's function, Neumann's function, and Green's function of the third kind, which is sometimes called the Robin function.

In the two-dimensional case, Green's function plays an important role in the theory of conformal mapping because it is conformally invariant. There is now a special type of Robin function which is likewise conformally invariant and leads to additional invariants and interesting applications. We shall define it as the function $R(z, \zeta)$ harmonic in the domain except for a logarithmic pole at ζ, and vanishing on a specified part of the boundary while its normal derivative vanishes on the rest of the boundary.

We begin the paper by recording some of the basic properties of the Robin function. We then establish its existence by displaying it as a solution to a certain extremal problem involving transfinite diameter. Next we use it to solve an extremal problem which arose in our previous study [5] of the energy functional of a multiply connected domain. We also apply it to find the sharp bounds of a quadratic form associated with the Riemann matrix. Finally, we discuss some further properties of the Robin function; for instance, we show that it