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TRACEABLE INTEGRAL KERNELS ON COUNTABLY GENERATED MEASURE SPACES

CHRIS BRISLAWN

Let K be a trace class operator on $L^2(X, \mathscr{M}, \mu)$ with integral kernel $K(x, y) \in L^2(X \times X, \mu \times \mu)$. An averaging process is used to define K on the diagonal in $X \times X$ so that the trace of K is equal to the integral of $K(x, x)$, generalizing results known previously for continuous kernels. This formula is also shown to hold for positive-definite Hilbert-Schmidt operators, thus giving necessary and sufficient conditions for the traceability of positive integral kernels. These results make use of Doob's maximal theorem for martingales and generalize previous results obtained by the author using Hardy-Littlewood maximal theory when $X \subset \mathbf{R}^n$.

1. Introduction. Let K be a compact operator on a Hilbert space, $\mathbf{H}$. The positive operator K^*K can be diagonalized by an orthonormal sequence $\{\psi_i\}_{i\in\mathbf{N}}$ of eigenvectors with corresponding eigenvalues $\mu_i > 0$. Define $\lambda_i = \sqrt{\mu_i}$ and $\phi_i = \lambda_i^{-1}K\psi_i$. The numbers λ_i, the eigenvalues of $|K|$, are called the *singular values* of K, and the sequence $\{\phi_i\}_{i\in\mathbf{N}}$ is also orthonormal. K is a Hilbert-Schmidt operator if the singular values are square-summable and a trace class operator if they are absolutely summable. Since $\ker(K) = \ker(K^*K)$, we have

$$\text{cl}\,\text{span}\{\psi_i\}_{i\in\mathbf{N}} = \ker(K^*K)^\perp = \ker(K)^\perp. \tag{1.1}$$

Let $\phi \otimes \overline{\psi}$ denote the rank 1 operator

$$\phi \otimes \overline{\psi} : \theta \mapsto (\theta, \psi)\phi.$$

By expanding $\theta \in \ker(K)^\perp$ in terms of the basis $\{\psi_i\}_{i\in\mathbf{N}}$, we see that

$$K\theta = \sum_{i=1}^\infty \lambda_i(\theta, \psi_i)\phi_i.$$

Thus, K has the canonical expansion

$$K = \sum_{i=1}^\infty \lambda_i \phi_i \otimes \overline{\psi}_i, \tag{1.2}$$

norm-convergent in $\mathscr{L}(H)$.