PACIFIC JOURNAL OF MATHEMATICS
Vol. 151, No. 1, 1991

HECKE EIGENFORMS AND REPRESENTATION NUMBERS OF QUADRATIC FORMS

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Using the theory of modular forms and Hecke operators, we obtain arithmetic relations on average representation numbers of positive definite quadratic forms in an even number of variables.

1. Introduction. When looking for multiplicative relations satisfied by representation numbers of quadratic forms, it is natural to study the effect of the Hecke operators on theta series attached to quadratic forms; in this paper we use such theta series to construct Hecke eigenforms and thereby obtain relations on weighted averages of representation numbers of quadratic forms.

When working over the rationals we obtain the relations

$$\mathbf{r}(\operatorname{gen} L, mm') = \lambda'(m)\mathbf{r}(\operatorname{gen} L', m') - \sum_{\substack{a|(m,m')\\ a>1}} \chi_L(a)a^{k-1}\mathbf{r}\left(\operatorname{gen} L, \frac{mm'}{a^2}\right)$$

where L is an even rank lattice equipped with a positive definite quadratic form, $\mathbf{r}(\operatorname{gen} L, n)$ is the average number of times the lattices in the genus of L represent n, $\lambda'(m)$ is an easily computed constant, χ_L is a quadratic character associated to L, and L' is a particular sublattice of L scaled by $1/m$ (see Theorem 3.9).

We assume herein that we are working with a totally positive quadratic form Q on vector space $\mathbf{V}$ of even dimension $2k$ over a totally real number field $\mathbf{K}$; for a lattice L on $\mathbf{V}$ we define the theta series attached to L to be the Hilbert modular form

$$\theta(L, \tau) = \sum_{x\in L} e^{\pi i \operatorname{Tr}(Q(x)\tau)}.$$

We observe that there is a family of lattices related to L which is partitioned into "nuclear families" such that the Hecke operators essentially permute the weighted averages of theta series attached to lattices within a nuclear family. Analyzing these permutations allows us to construct Hecke eigenforms, and analyzing the behavior of these