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MULTIPLE HARMONIC SERIES

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We consider several identities involving the multiple harmonic series

$$\sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}},$$

which converge when the exponents i_j are at least 1 and $i_1 > 1$. There is a simple relation of these series with products of Riemann zeta functions (the case $k = 1$) when all the i_j exceed 1. There are also two plausible identities concerning these series for integer exponents, which we call the sum and duality conjectures. Both generalize identities first proved by Euler. We give a partial proof of the duality conjecture, which coincides with the sum conjecture in one family of cases. We also prove all cases of the sum and duality conjectures when the sum of the exponents is at most 6.

1. Introduction. The problem of computing the doubly infinite series

$$\sum_{n_1\geq n_2\geq 1} \frac{1}{n_1^a n_2^b}, \tag{1}$$

which converges when $a > 1$ and $b \geq 1$, was discussed by Euler and Goldbach in their correspondence of 1742–3 [**3**]. Euler evaluated several special cases of (1) in terms of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

Later, in a paper of 1775 [**2**], Euler found a general formula for (1) in terms of the zeta function when a and b are positive integers whose sum is odd. The simplest such result is

$$\sum_{n_1\geq n_2\geq 1} \frac{1}{n_1^2 n_2} = 2\zeta(3), \tag{2}$$

which has been rediscovered many times since (see [**1**, p. 252] and the references cited there).

We shall consider multiple series of the form

$$S(i_1, i_2, \ldots, i_k) = \sum_{n_1\geq n_2\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}$$