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THE BRAID INDEX OF GENERALIZED CABLES

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If one knot is fashioned into another, by replacing each strand with q strands, then *something* gets multiplied by q. What? The answer should not be overly dependent on how these strands are intertwined. We show that an invariant called the *braid index* is an answer. This proposition is apparently new. Another answer covered by our proof is the *bridge number*, though this was proved by Shubert in 1954. It was only with the advent of the Jones polynomial and its relatives in the mid 1980s, that much attention has been given to the braid index. For example, the knots obtained by repeated *period doubling* were shown to obey the multiplication rule, though no one seems to have thought of it this way. Their braid indices are powers of 2. We first considered the current proposition in trying to show that a certain knot, known to have braid index 5, could not be a two-cabling of anything.

DEFINITIONS. It is a classical result of Alexander [**A**] that any link, that is, a finite collection of smooth oriented simple closed curves embedded in Euclidian 3-space, can be isotoped into the (closure) of a braid on some number of strands, say n. By the *braid index* of a link we mean the least such number n.

The *bridge number* is the minimal number of local maxima for any smooth isotopic copy of a link L. See Shubert [**S1**; Satz 9, p. 283]. Our result follows as a corollary to Shubert's theorem in those cases for which these invariants are equal, since the bridge number is trivially seen to be less than or equal to the braid index.

We use $b(L)$ to denote the braid index (respectively bridge number) of an oriented link L. In the theorem below, we assume that each component of our link is knotted; this assumption is necessary in that (for example) any (p, q) torus link is both a p-fold and a q-fold cabling of the unknot. For $p \neq q$ its braid index cannot be both p and q. In fact its braid index is the lesser of p and q as can be seen, e.g., by the theorem of Morton [**M**] and Franks-Williams [**FW**]. That is, if L is the closure of a positive braid on p strands which has a full twist, then L has braid index p. But, any (p, q) torus link is a positive braid on p strands as follows: denote the generators of the braid group B_p by $2, 3, \dots, p$. The (p, q) torus link L_{pq} is the closure of the braid $\beta = (2, 3, \dots, p)^q$. Since the full twist