PACIFIC JOURNAL OF MATHEMATICS
Vol. 155, No. 2, 1992

INVARIANT SUBSPACES AND HARMONIC CONJUGATION ON COMPACT ABELIAN GROUPS

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Let Γ be a dense subgroup of the real line $\mathbb{R}$. Endow Γ with the discrete topology and the order it inherits from $\mathbb{R}$, and let K be the dual group of Γ. Helson's classic theory of generalized analyticity uses the spectral decomposability of unitary groups to establish a one-to-one correspondence between the cocycles on K and the normalized simply invariant subspaces of $L^2(K)$. This theory has been extended to the invariant subspaces of $L^p(K)$, $1 < p < \infty$, by using recent results concerning the spectral decomposability of uniformly bounded one-parameter groups acting on UMD spaces. We show here that each cocycle A on K can be used to transfer the classical Hilbert transform from $L^1(\mathbb{R})$ to $L^1(K)$ in terms of almost everywhere convergence on K so that in the interesting case (i.e., when A is not a coboundary) the corresponding invariant subspace of $L^p(K)$ is a generalized ergodic Hardy space. This description of the invariant subspaces explicitly identifies the role of the Hilbert transform in generalized analyticity on K. The formulation in terms of almost everywhere convergence on K provides an intrinsic viewpoint which extends to the case $p = 1$.

1. Cocycles and invariant subspaces. Throughout what follows K will be a compact abelian group other than $\{0\}$ or the unit circle $\mathbb{T}$ such that the dual group of K is archimedean ordered. Equivalently, we shall require K to be the dual group of Γ, where Γ arises as a dense subgroup of the additive real line $\mathbb{R}$, and Γ is then endowed with the natural order of $\mathbb{R}$ and the discrete topology. For each $\lambda \in \Gamma$ we denote by χ_λ the corresponding character on K (evaluation at λ), and for each $t \in \mathbb{R}$ we let e_t be the element of K defined by $e_t(\lambda) = e^{it\lambda}$ for all $\lambda \in \Gamma$. As is well known, $t \to e_t$ is a continuous isomorphism of $\mathbb{R}$ onto a dense subgroup of K. For $1 \le p < \infty$ we follow Helson [12] in defining a *simply invariant subspace* of $L^p(K)$ to be a closed subspace M of $L^p(K)$ such that $\chi_\lambda M \subseteq M$ for all $\lambda > 0$, but for some $\alpha < 0$, $\chi_\alpha M$ is not a subset of M. A simply invariant subspace M of $L^p(K)$ is said to be *normalized* provided $M = \bigcap\{\chi_\lambda M : \lambda \in \Gamma,\ \lambda < 0\}$. The set of all normalized simply invariant subspaces of $L^p(K)$ will be denoted by $\mathscr{S}_p$. A *cocycle* on