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BORDISM AND REGULAR HOMOTOPY OF LOW-DIMENSIONAL IMMERSIONS

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In this paper we study the geometric characteristics of low-dimensional immersions. Smale asked, in his paper on immersions of the k-sphere in R^n, what are explicit generators for the groups of regular homotopy classes of immersions? We answer this for the 3-sphere in R^4 and R^5. For S^3 in R^4, the answer is:

THEOREM. ***The standard (Froissart-Morin) eversion of S^2 in R^3 has, as a track, an immersion of $S^2 \times I$ in R^4 whose ends are embedded S^2s. Each of these bounds a 3-ball in R^4. Capping off the track with these 3-balls yields an immersion $K: S^3 \to R^4$. Performing the eversion twice and capping off gives an immersion $E: S^3 \to R^4$. The immersions E and K generate the group of regular homotopy classes of immersions of S^3 in R^4.***

We also relate the invariants of an immersion which bounds an immersion of a manifold of one higher dimension to the characteristic classes of that manifold.

1. Notation, definitions, and preliminary results.

We begin by establishing our notation and definitions. All manifolds and maps in this paper are smooth, unless otherwise noted.

DEFINITION. A *k-frame* in a vector space is an ordered k-tuple of linearly independent vectors $v^1, \dots, v^k$. A k-frame in a k-dimensional space may be considered an ordered basis of the space. Frames are denoted by square brackets, and we implicitly extend any maximal rank linear map between vector spaces V and W to also take frames in V to frames in W, by defining

$$T[v^1, \dots, v^k] = [Tv^1, \dots, Tv^k].$$

DEFINITION. $V_{n,k}$ denotes the Stiefel manifold of k-frames in R^n. If N is an n-plane bundle, $V_k(N)$ denotes the associated bundle of k-frames.

NOTATION. TM denotes the *tangent bundle* of the manifold M. The bundle projection is denoted by π. For a typical point p of M, the *fiber over* p, $\pi^{-1}(p)$, is denoted by T_pM.