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SOME NUMERIC RESULTS ON ROOT SYSTEMS

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Let Φ be an irreducible root system (sometimes we denote Φ by $\Phi(X)$ to indicate its type X). Choose a simple root system Π in Φ. Let Φ^+ (resp. Φ^-) be the corresponding positive (resp. negative) root system of Φ. By a subsystem Φ' of Φ (resp. of Φ^+), we mean that Φ' is a subset of Φ (resp. of Φ^+) which itself forms a root system (resp. a positive root system). We refer the readers to Bourbaki's book for the detailed information about root systems. Among all subsystems of Φ, the subsystems of Φ of rank 2 and of type $\neq A_1 \times A_1$ are of particular importance in the theory of Weyl groups and affine Weyl groups (see the papers by Jian-yi Shi). In the present paper, we shall compute the number of such subsystems of Φ for an irreducible root system Φ of any type. Some interesting properties of Φ are also obtained.

1. The number $h(\alpha)$. Let $\langle\ ,\ \rangle$ be an inner product of the euclidean space E spanned by Φ. For any $\alpha \in \Phi$, we denote by $|\alpha|$ the length of α, by $\alpha^\vee$ the dual root $2\alpha/\langle\alpha, \alpha\rangle$ of α and by s_α the reflection in E which sends any vector $v \in E$ to $s_\alpha(v) = v - \langle v, \alpha^\vee\rangle\alpha$. For $\alpha, \beta \in \Phi$, we write $\alpha < \beta$ if $\beta - \alpha$ is a sum of some positive roots.

For $\alpha \in \Phi$, we define the sets $D(\alpha) = \{\beta \in \Phi \mid \alpha + \beta \in \Phi\}$, $D^+(\alpha) = D(\alpha) \cap \Phi^+$ and $D^-(\alpha) = D(\alpha) \cap \Phi^-$. Let $d(\alpha)$ be the cardinality of the set $D^+(\alpha)$. Also, we denote by $\mathrm{ht}(\alpha)$ the height of α, i.e. $\mathrm{ht}(\alpha) = \sum_{\beta\in\Pi} a_\beta$ if $\alpha = \sum_{\beta\in\Pi} a_\beta \beta$ with $a_\beta \in \mathbb{Z}$.

For any $\alpha \in \Phi^+$, there exists a sequence ξ of roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r$ in Φ^+ such that $\alpha_r \in \Pi$ and for every i, $1 < i \leq r$, we have $\alpha_{i-1} > \alpha_i = s_{\delta_i}(\alpha_{i-1})$ for some $\delta_i \in \Pi$. Such a sequence ξ is called a root path from α to Π. We denote by $h(\alpha, \xi)$ the length r of ξ. We shall deduce a formula for the number $h(\alpha, \xi)$, from which we shall see that $h(\alpha, \xi)$ is actually independent on the choice of a root path ξ from α to Π but only dependent on the root α.

Note that if the root system Φ contains roots of two different lengths and if $\alpha = \sum_{\beta\in\Pi} a_\beta \beta$ is a long root of Φ with $a_\beta \in \mathbb{Z}$ then each coefficient a_β with β short is divisible by $|\alpha|^2/|\beta|^2$.

LEMMA 1.1. *Let $\alpha = \sum_{\beta\in\Pi} a_\beta \beta$, $a_\beta \in \mathbb{Z}$, be a root of Φ^+ and let ξ be a root path from α to Π. Then*