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MAPS ON INFRA-NILMANIFOLDS

–Rigidity and applications to Fixed-point Theory

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We show that Bieberbach's rigidity theorem for flat manifolds still holds true for any continuous maps on infra-nilmanifolds. Namely, every endomorphism of an almost crystallographic group is semi-conjugate to an affine endomorphism. Applying this result to Fixed-point theory, we obtain a criterion for the Lefschetz number and Nielsen number for a map on infra-nilmanifolds to be equal.

0. Infra-nilmanifolds. Let G be a connected Lie group. Consider the semi-group $\mathrm{Endo}(G)$, the set of all endomorphisms of G, under the composition as operation. We form the semi-direct product $G \rtimes \mathrm{Endo}(G)$ and call it $\mathrm{aff}(G)$. With the binary operation

$$(a, A)(b, B) = (a \cdot Ab, AB),$$

the set $\mathrm{aff}(G)$ forms a semi-group with identity $(e, 1)$, where $e \in G$ and $1 \in \mathrm{Endo}(G)$ are the identity elements. The semi-group $\mathrm{aff}(G)$ "acts" on G by

$$(a, A) \cdot x = a \cdot Ax.$$

Note that (a, A) is not a homeomorphism unless $A \in \mathrm{Aut}(G)$. Clearly, $\mathrm{aff}(G)$ is a subsemi-group of the semi-group of all continuous maps of G into itself, for $((a, A)(b, B))x = (a, A)((b, B)x)$ for all $x \in G$. We call elements of $\mathrm{aff}(G)$ *affine endomorphisms.*

Suppose G is a connected, simply connected, nilpotent Lie group; $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ is called the group of affine automorphisms of G. Let $\pi \subset \mathrm{Aff}(G)$ be a discrete subgroup such that $\Gamma = \pi \cap G$ has finite index in π. Then $\pi \backslash G$ is compact if and only if Γ is a lattice of G. In this case, π is called an *almost crystallographic group.* If moreover, π is torsion-free, π is an *almost Bieberbach group.* Such a group is the fundamental group of an infra-nilmanifold. According to Gromov and Farrell-Hsiang, the class of infra-nilmanifolds coincides with the class of almost flat manifolds.