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INVARIANT THEORY OF SPECIAL ORTHOGONAL GROUPS

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In this paper we study the action of $\mathrm{SO}(n)$ on m-tuples of $n \times n$ matrices by simultaneous conjugation. We show that the polynomial invariants are generated by traces and polarized Pfaffians of skewsymmetric projections. We also discuss the same problem for other classical groups.

1. Special orthogonal groups. Let F be a field of characteristic 0. If A is a skewsymmetric $2k \times 2k$ matrix over F, we denote the Pfaffian of A by $\operatorname{pf} A$. It satisfies $\det A = \operatorname{pf}^2 A$ and $\operatorname{pf}(gAg^t) = \det g \operatorname{pf} A$. For an arbitrary $2k \times 2k$ matrix M, we define $\widetilde{\operatorname{pf}}(M) = \operatorname{pf}(M - M^t)$ to be the Pfaffian of the skewsymmetric projection of M. This is clearly an $\mathrm{SO}(2k, F)$ invariant. By abuse of notation we will refer to $\widetilde{\operatorname{pf}}$ as the Pfaffian, too.

Let $W = W(n, m, F)$ be the vector space of m-tuples of $n \times n$ matrices over F on which a group $G \subset \mathrm{GL}(n, F)$ acts by simultaneous conjugation. For $G = \mathrm{SO}(2, F)$,the invariants $P[W(2, m, F)]^G$ were determined in [**1**]. They are generated by the invariants $\operatorname{tr} P(A, A^t)$ and $\widetilde{\operatorname{pf}} P(A, A^t)$ where $A \in W(2, m, F)$ and P is noncommutative polynomial.

We will see that for n odd we do not get any more invariants when we restrict $O(n, F)$ to $\mathrm{SO}(n, F)$. In the even case we have the following crucial Lemma that indicates why Pfaffian appears.

LEMMA 1. *Let $x_1, \ldots, x_{2k} \in F^{2k}$ and let $[x_1, \ldots, x_{2k}]$ denote the determinant of the matrix with columns $x_1, \ldots, x_{2k}$. Then*

$$[x_1, \ldots, x_{2k}] = \widetilde{\operatorname{pf}}(x_1 x_2^t + \cdots + x_{2k-1} x_{2k}^t).$$