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DIAGONALIZING HILBERT CUSP FORMS

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We develop an operator $C_{\mathfrak{q}}(\Psi_Q)$ on the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ of Hilbert cuspforms as an alternative to the Hecke operator $T_{\mathfrak{q}}$ for primes $\mathfrak{q}$ dividing $\mathcal{N}$. For $\mathbf{f} \in \mathcal{S}_k(\mathcal{N}, \Psi)$ a newform, we have $\mathbf{f} \mid C_{\mathfrak{q}}(\Psi_Q) = \mathbf{f} \mid T_{\mathfrak{q}}$. We are able to decompose the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ into a direct sum of common eigenspaces of $\{T_{\mathfrak{p}},\ C_{\mathfrak{q}}(\Psi_Q) :\ \mathfrak{p} \nmid \mathcal{N},\ \mathfrak{q} \mid \mathcal{N}\}$, each of dimension one. Each common eigenspace is spanned by an element with the property that its eigenvalue with respect to $T_{\mathfrak{p}}$ (resp. $C_{\mathfrak{q}}(\Psi_Q)$) is its $\mathfrak{p}^{\text{th}}$ (resp $\mathfrak{q}^{\text{th}}$) Fourier coefficient. We finish by deriving bounds for the eigenvalues of $C_{\mathfrak{q}}(\Psi_Q)$.

Introduction. Let $\mathcal{S}_k(\mathcal{N}, \Psi)$ denote the space of Hilbert cusp forms of Hecke character Ψ. Shemanske and Walling [**7**] characterized the newform theory for $\mathcal{S}_k(\mathcal{N}, \Psi)$ which is analogous to that derived in [**1**] for the elliptic modular case. They decompose the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ into a direct sum of common eigenspaces for the Hecke operators $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$. The non-zero elements of the one-dimensional common eigenspaces are called newforms, and a newform can be normalized such that its $\mathfrak{p}^{\text{th}}$ Fourier coefficient is equal to its eigenvalue for $T_{\mathfrak{p}}$. They also show that each common eigenspace of $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$ has a basis of the form $\{\mathbf{g} \mid B_{\mathfrak{L}} : \mathbf{g} \in \mathcal{S}_k(\mathcal{M}, \Psi)$ a newform , $\mathcal{M} \mid \mathcal{N}, \mathfrak{L} \mid \mathcal{N}\mathcal{M}^{-1}\}$. While the Hecke operators $\{T_{\mathfrak{q}} : \mathfrak{q} \mid \mathcal{N}\}$ act invariantly on these eigenspaces, there generally does not exist a basis for these eigenspaces which consists of eigenforms for $\{T_{\mathfrak{q}} : \mathfrak{q} \mid \mathcal{N}\}$.

In this work, we resolve this particular difficulty by replacing $T_{\mathfrak{q}}$, $\mathfrak{q} \mid \mathcal{N}$ by the operator $C_{\mathfrak{q}}(\Psi_Q)$. It is defined using the Hecke operator $T_{\mathfrak{q}}$ and the Hilbert analog of the Atkin-Lehner W_Q operator of [**7**], and hence depends upon a choice of Hecke character Ψ_Q. We are able to diagonalize the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ with respect to the family $\{T_{\mathfrak{p}},\ C_{\mathfrak{q}}(\Psi_Q) :\ \mathfrak{p} \nmid \mathcal{N},\ \mathfrak{q} \mid \mathcal{N}\}$. Further, we are able to establish that each common eigenspace is one-dimensional and is spanned by a form whose $\mathfrak{p}$th (resp $\mathfrak{q}^{\text{th}}$) Fourier coefficient is its eigenvalue with