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CONVERGENCE OF INFINITE EXPONENTIALS

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In this paper we give two tests of convergence for an infinite exponential $a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot}}}}}$. We also show that these tests are essentially the best possible.

1. Introduction and Statement of Results. Given a sequence of positive real numbers a_n, $n = 1, 2, 3, \ldots$, we associate with it a sequence of partial exponentials E_n, $n = 1, 2, 3, \ldots$, defined by

$$E_n = a_1^{a_2^{\cdot^{\cdot^{a_n}}}}. \tag{1.1}$$

We will call $\{a_n\}$ a sequence of exponents and the sequence $\{E_n\}$ an infinite exponential. As in the study of sums and products one would like to develop tests of convergence of an infinite exponential. Euler [**E**] was the first to give such a test. He showed that in the special case $a_1 = a_2 = a_3 = \cdots = a$, E_n is convergent if and only if $e^{-e} \leq a \leq e^{1/e}$. This result has been rediscovered by many authors. An extensive bibliography of papers containing this and related results may be found in the survey paper by Knoebel [**K**].

In the general case of non-constant exponents the best known results are due to Barrow [**B**]. He showed (although some of his arguments are rather sketchy) that $\{E_n\}$ is convergent for $e^{-e} \leq a_n \leq e^{1/e}$, $n \geq n_0$. He also considered the cases $a_n \geq e^{1/e}$ and $a_n \leq e^{-e}$. In the first case, writing $a_n = e^{1/e} + \epsilon_n$, with $\epsilon_n \geq 0$, he showed that $\{E_n\}$ is convergent if

$$\lim_{n\to\infty} \epsilon_n n^2 < \frac{e^{1/e}}{2e}, \tag{1.2}$$

and is divergent if

$$\lim_{n\to\infty} \epsilon_n n^2 > \frac{e^{1/e}}{2e}. \tag{1.3}$$