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A FROBENIUS PROBLEM ON THE KNOT SPACE

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According to J.-L. Brylinski, there is a natural almost complex structure J on the space K of all knots in the Euclidean space R^3. The almost complex structure is formally integrable on K, i.e, the Nijenhuis tensor of J vanishes. The problem is whether J is integrable and hence K is a complex manifold. In this paper, we study the integrability of J explicitly in view point of a Frobenius problem.

1. Introduction

A knot is by definition a smooth imbedded circle in the Euclidean space R^3. The knot space is the space of all knots. In this paper, we study an integrability problem on the knot space which is as follows: According to Brylinski [**3**, **4**], for any $\gamma \in K$, the tangent space $T_\gamma K$ is the space of sections of the normal bundle of γ in R^3. A natural almost complex structure J is defined on K as a rotation of $\frac{\pi}{2}$ in the normal plane bundle. J is formally integrable on K, i.e, the Nijenhuis tensor of J vanishes. Compared to the well-known theorem of Newlander-Nirenberg [**17**], the problem is whether J is integrable and hence K is a complex manifold.

A result of Drinfeld and LeBrun [**3**, **4**] is that J is weakly integrable on the space K_0 of real analytic knots, i.e., there are enough holomorphic functions on each local chart of K_0. In Lempert [**15**], the theory of twistor CR-manifolds is used to prove that J is weakly integrable on the space of real analytic knots in a real analytic 3-manifold with a real analytic metric. It is also proved that J is not integrable on the space K and K_0, i.e., there is no open set $U \neq \phi$ on the knot space which is biholomorphic to an open set in $T_\gamma K$ or $T_\gamma K_0$. LeBrun [**14**] has a similar result on the so-called space of world-sheets which are time-like 2-surfaces in 4-manifold with a Lorentzian metric.

In this paper, we define a natural local coordinate system on K and study the integrability of J explicitly in view point of a Frobenius problem. It will be shown that in the local coordinate system J can be written explicitly to see that it is real analytic and the $\bar{\partial}$-equation can be complexified to obtain a Frobenius problem and the Frobenius problem can be further reduced to a first order nonlinear partial differential equation in two dimensions. In the