PACIFIC JOURNAL OF MATHEMATICS
Vol. 172, No. 1, 1996

THE QUASI-LINEARITY PROBLEM FOR C^*-ALGEBRAS

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Let $\mathcal{A}$ be a C^*-algebra with no quotient isomorphic to the algebra of all two-by-two matrices. Let μ be a quasi-linear functional on $\mathcal{A}$. Then μ is linear if, and only if, the restriction of μ to the closed unit ball of $\mathcal{A}$ is uniformly weakly continuous.

Introduction.

Throughout this paper, $\mathcal{A}$ will be a C^*-algebra and A will be the real Banach space of self-adjoint elements of $\mathcal{A}$. The unit ball of A is A_1 and the unit ball of $\mathcal{A}$ is $\mathcal{A}_1$. We do not assume the existence of a unit in $\mathcal{A}$.

Definition. A *quasi-linear functional* on A is a function $\mu : A \to \mathbb{R}$ such that, whenever B is an abelian subalgebra of A, the restriction of μ to B is linear. Furthermore μ is required to be bounded on the closed unit ball of A.

Given any quasi-linear functional μ on A we may extend it to $\mathcal{A}$ by defining

$$\tilde{\mu}(x + iy) = \mu(x) + i\mu(y)$$

whenever $x \in A$ and $y \in A$. Then $\tilde{\mu}$ will be linear on each maximal abelian *-subalgebra of $\mathcal{A}$. We shall abuse our notation by writing 'μ' instead of '$\tilde{\mu}$'.

When $\mathcal{A} = \mathrm{M}_2(\mathbb{C})$, the C^* -algebra of all two-by-two matrices over $\mathbb{C}$, there exist examples of quasi-linear functionals on $\mathcal{A}$ which are not linear.

Definition. A local *quasi-linear functional* on A is a function $\mu : A \to \mathbb{R}$ such that, for each x in A, μ is linear on the smallest norm closed subalgebra of A containing x. Furthermore μ is required to be bounded on the closed unit ball of A.

Clearly each quasi-linear functional on A is a local quasi-linear functional. Surprisingly, the converse is false, even when A is abelian (see Aarnes [**2**]). However when A has a rich supply of projections (e.g. when $\mathcal{A}$ is a von Neumann algebra) each local quasi-linear functional is quasi-linear [**3**].

The solution of the Mackey-Gleason Problem shows that every quasi-linear functional on a von Neumann algebra $\mathcal{M}$, where $\mathcal{M}$ has no direct summand of Type I_2, is linear [**4**, **5**, **6**]. This was first established for positive quasi-linear functionals by the conjunction of the work of Christensen [**7**] and