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THE COVERS OF A NOETHERIAN MODULE

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In this paper we define the covers of a module and describe some of their applications.

1. Introduction.

Let R be a commutative ring and A an R-module. A cover of A is defined to be a subset T of $\operatorname{Max}(R)$ satisfying that for any $x \in A$, $x \neq 0$, there is $M \in T$ such that $0 :_R x \subseteq M$. If we denote by J the intersection of all the maximal ideals belonging to T and suppose that $A \neq 0$ is finitely generated, then we have $JA \neq A$. This generalises the Nakayama's lemma; if, in addition, R is Noetherian, then $\bigcap\limits_{n=1}^{\infty} J^n A = 0$. This is a generalization of a well-known result. A key observation for the covers is that, in the case that R is Noetherian and A is finitely generated, there is a cover T of A which is itself a finite set. From this we have the following result: Let R be a Noetherian ring. Then there is a finite number of maximal ideals $M_1, \dots, M_m$ of R such that $\bigcap\limits_{n=1}^{\infty} J^n = 0$, where $J = \bigcap\limits_{i=1}^{m} M_i$. This generalises the Krull's theorem for Jacobson radicals. Using this result we can embed the Noetherian ring R in the J-adic completion $\widehat{R}$ of R, which is a complete semi-local Noetherian ring; besides, if R is a Cohen-Macaulay (C-M for short) ring, then $\widehat{R}$ is a C-M ring. We also use the covers to deal with the maximal component of a finitely generated module over a Noetherian ring, which was introduced by Matlis in [**3**].

Throughout the paper, R will denote a (non-trivial) commutative ring with identity. Also, if T is a subset of $\operatorname{Max}(R)$ we denote by $\cap T$ (resp. $\cup T$) the intersection (resp. union) of all the maximal ideals belonging to T.

2. The covers.

In this section we define the covers of a module and generalise some known results.

Definition. Let A be an R-module. A subset T of $\operatorname{Max}(R)$ is called a cover of A if for any $x \in A$, $x \neq 0$, there is $M \in T$ such that $0 :_R x \subseteq M$.

Clearly, if T is a cover of A and B is a submodule of A, then T is a cover of B. If T is a cover of A and $T \subseteq T' \subseteq \operatorname{Max}(R)$, then T' is a cover of A. We