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LINEAR COMBINATIONS OF LOGARITHMIC DERIVATIVES OF ENTIRE FUNCTIONS WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Let $F_1, F_2, \ldots, F_L$ be entire functions of finite order and let $c_1, c_2, \ldots, c_L$ be complex numbers whose convex hull does not contain 0. A lower bound in terms of the counting functions of the zeros of the F_j's is obtained for

$$\left|\sum_{j=1}^{L} c_j re^{i\theta} F_j'(re^{i\theta})/F_j(re^{i\theta})\right|$$

valid for r in a set of positive logarithmic density and θ in a set $U_r \subset [0, 2\pi]$ of fixed positive measure. This bound is used to extend a result of Bank and Langley concerning the exponent of convergence of the zero sequences of solutions of certain linear differential equations with entire coefficients.

1. Introduction.

In this paper we are concerned with the behavior of the logarithmic derivative F'/F of an entire function F of finite order, and most particularly with lower bounds for $|F'/F|$. From the argument principle it follows that if F has no zeros on $|z| = r$, then

$$\frac{1}{2\pi}\int_0^{2\pi} re^{i\theta}\frac{F'(re^{i\theta})}{F(re^{i\theta})}\, d\theta = n(r, 0, F),$$

where $n(r, 0, F)$ denotes the number of zeros of F in $|z| \leq r$ counting multiplicity. Our principal result implies for most values of r that if $0 < \beta < 1$, then the modulus of the above integrand is greater than or equal to $\beta n(r, 0, F)$ on a substantial portion of the circle $|z| = r$.

For applications to the behavior of solutions of certain differential equations it is useful to formulate our result in terms of linear combinations of logarithmic derivatives.

Theorem 1. *Suppose F_j, $1 \leq j \leq L$, are entire functions each with order not exceeding $\rho < \infty$. Suppose c_j, $1 \leq j \leq L$, are complex numbers lying in a*