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CHERN CLASSES OF VECTOR BUNDLES ON ARITHMETIC VARIETIES

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Let $\overline{F}$ be a Hermitian vector bundle on an arithmetic variety X over $\mathbb{Z}$. We prove an inequality between the L^2-norm of an element in $H^1(X, F^\vee)$ and arithmetic Chern classes of $\overline{F}$ under certain stability condition. This is a higher dimensional analogue of a result of C. Soulé for Hermitian line bundles on arithmetic surfaces. We observe that our result is related to a conjectural inequality of Miyaoka-Yau type.

Introduction.

In a recent paper [**S**], C. Soulé obtained an analogue for arithmetic surfaces of Kodaira-Ramanujam vanishing theorem. Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let X be an arithmetic surface over $\mathcal{O}_K$ with the smooth, geometrically irreducible generic fiber and $\overline{L} = (L, h)$ a Hermitian line bundle on X. We denote by $\hat{c}_1(\overline{L})$ its arithmetic first Chern class. The main result in *loc.cit.* states that if $\overline{L}$ satisfies certain positivity assumption, then there exist explicit constants A, B such that for any non-torsion element $e \in H^1(X, L^\vee)$, we have

$$\hat{c}_1(\overline{L})^2 \leq A \log \|e\| + B.$$

Here $\|e\|$ denotes the supremum of the L^2-norm $\|\sigma(e)\|_{L^2}$ when σ runs over all infinite places of K.

In this paper we prove a similar inequality for Chern classes of certain Hermitian vector bundles on higher dimensional arithmetic varieties over $\mathbb{Z}$. For this purpose, we need the assumption that these bundles are stable with respect to an arithmetically ample line bundle. Unfortunately we have to put further rather restrictive assumptions on their first Chern classes (see §1. for a precise statement).

There are some applications of our main result. First we obtain an arithmetic vanishing of the first cohomology groups $H^1(X, F^\vee)$: It is possible to bound the number of elements of L^2-norm less than or equal to one by certain constants. Secondly we prove an inequality for Chern classes of arithmetic varieties with arithmetically ample relative canonical bundle $\overline{\omega}_{X/\mathbb{Z}}$, which can be considered as an arithmetic analogue of Miyaoka-Yau inequality ([**Y**]).